Balancing Load via Small Coalitions in Selfish Ring Routing Game

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Abstract

This paper concerns the asymmetric atomic selfish routing game for load balancing in ring networks. In the selfish routing, each player selects a path in the ring network to route one unit traffic between its source and destination nodes, aiming at a minimum maximum link load along its own path. The selfish path selections by individuals ignore the system objective of minimizing the maximum load over all network links. This selfish ring load (SRL) game arises in a wide variety of applications in decentralized network routing, where network performance is often measured by the price of anarchy (PoA), the worst-case ratio between the maximum link loads in an equilibrium routing and an optimal routing. It has been known that the PoA of SRL with respect to classical Nash equilibrium cannot be upper bounded by any constant, showing large loss of efficiency at some Nash equilibrium outcome.

In an effort to improve the network performance in the SRL game, we generalize the model to so-called SRL with collusion (SRLC) which allows coordination within any coalition of up to \( k \) selfish players on the condition that every player of the coalition benefits from the coordination. We prove that, for \( m \)-player game on \( n \)-node ring, the PoA of SRLC is \( n - 1 \) when \( k \leq 2 \), drops to 2 when \( k = 3 \), and is at least \( 1 + 2/m \) for \( k \geq 4 \). Our study shows that on one hand, the performance of ring networks, in terms of maximum load, benefits significantly from coordination of self-interested players within small-sized coalitions; on the other hand, the equilibrium routing in SRL might not reach global optimum even if any number of players can coordinate.

Keywords  Selfish routing, bottleneck congestion game, load balancing, ring networks, \( k \)-strong equilibrium, price of anarchy

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1 Introduction

For decades, it has been the responsibility of the network central authority to route traffic and all network participants are assumed to obey the protocol. However, modern networks usually operate at a scale that makes the use of centralized protocols challenging. Thus, recent trends in the design and analysis of network routing take into account the rational behaviors of selfish network users. Among many others, selfish routing (Roughgarden and Tardos 2002) models network routing from a game-theoretic perspective, in which network users are viewed as self-interested strategic players participating in a competitive game. Each player, with his own pair of source and destination nodes in the network, aims to establish a communication path (between his source and destination) along which he experiences latency or bottleneck congestion as low as possible. This paper concerns with the latter objective, and studies selfish routing in ring networks on maximum load (referred to as SRL) in which both individual players and the central authority wish to minimize their own maximum link loads. Our study focuses on ring networks since ring has been a fundamental topology frequently encountered in communication networks, and attracted considerable attention and efforts from the research community (Muñoz et al. 2011; Bentza et al. 2009; Anshelevich and Zhang 2008; Chen et al. 2010; Cheng 2004; Wang 2005; Blum et al. 2001; Schrijver et al. 1998).

Our study on SRL is inspired by the end-to-end packet delivery in communication networks, where each packet is to be delivered along a path greedily selected by its corresponding player without considering the system-wide criteria. In heavily congested networks, the delay of a packet is governed by the bottleneck congestion (the maximum link load) the packet experiences (Busch and Magdon-Ismail 2009). From a systemic perspective, the performance of a communication network is closely related to the performance of its most congested link (Banner and Orda 2007; Cole et al. 2006; Qiu et al. 2006), especially when robustness to bursty traffic (Banerjee and Yoo 1997) or to growing demand (Wang and Wang 1999) is a priority.

In the absence of a central authority which can impose and maintain globally efficient routing strategies on network traffic, network designers are often interested in a stable outcome that is as close to the system optimum as possible. The most popular solution concept of Nash Equilibrium (NE) refers to the stable state from which no individual would deviate unilaterally. Given a certain social cost that measures the network performance (e.g., the overall maximum link load in SRL), the efficiency of NE is often quantified by Price of Anarchy (PoA) and Price of Stability (PoS), which are the worst-case ratio and the best-case ratio, respectively, between the social costs in a NE and in a globally optimal solution (Koutsoupias and Papadimitriou 1999; Anshelevich et al. 2009). Although the selfish routing in general networks on maximum load (referred to as SL) always admits NE, its PoA can grow linearly with the size of network; the known worst case of SL appears in ring networks (Busch and Magdon-Ismail 2009). This suggests a natural starting point – SRL, for improving SL games. Good resolution for SRL may provide insights for pursuing nice outcomes in SL.

While NE is a powerful tool for predicting outcomes in competitive environments, its notion of stability applies only to unilateral deviations under the assumption that users (players) are completely non-cooperative, isolated entities in networks (games), acting not only selfishly but also independently. However, in numerous competitive situations, given today’s communication infrastructure, a group of selfish users may and does coordinate a joint deviation if it is profitable to all the members of the group (Procaccia and Rosenschein 2006): businesses agree to cooperate for mutual benefits, and agents contract bilaterally or multilaterally to take joint actions for common efficiency (Arcaute et al. 2009). In these more realistic situations that allow some level of coordination, the NE is not necessarily sustainable in that it may not reflect
rational behaviors of players. To address the issue of coordination in competitive games, Aumann (1959) introduced the concept of *Strong Equilibrium* (SE), which ensures stability against deviations by *every* possible coalition of players, namely, no coalition of any size can cooperatively deviate in a way that *strictly* benefits *all* the group members, taking the actions of the players outside the coalition as given. The resilience to deviations by all coalitions turns out a very robust and appealing notion of stability, motivating extensive research on SE (Albers 2009; Andelman et al. 2009; Feldman and Tamir 2009; Fiat et al. 2007; Holzman and Law-yone 2003); but this stronger equilibrium concept suffers from a major criticism: it is too strong in the sense that SE rarely exists. Even if in some cases we can prove SE exists in theory, or even more luckily, find a SE in polynomial time, it is usually very hard for a distributed system to reach such a strong equilibrium by self-organized players. As pointed out by Hayrapetyan et al. (2006), players may not all cooperate for the greater good because of communication and/or computation difficulty in large networks (see also Andelman et al. 2009; Jackson 2005; Leonardi and Sankowski 2007). On the other hand, we must recognize that, in many cases, players will form small coalitions to improve their well-being (Hayrapetyan et al. 2006). Unfortunately, in some settings small coalitions do not necessarily help a lot, or even worse, harm the social welfare (Hayrapetyan et al. 2006; Leonardi and Sankowski 2007) in comparison with the corresponding coalition-free game. In this paper we study *SRL with Collusion* (SRLC), referring to the SRL game that allows dynamic coalitional cooperations. Our results show that *small-sized* coalitions do improve the network performance in SRL *significantly*, leading to outcomes with nearly optimal efficiency. Without bothering with exponentially many possible coalitions as a SE has to consider, the SRL with small coalition is easy to form and therefore very practical from an algorithmic point of view.

*Related work* Our SRLC model generalizes the SRL game, which is the restriction of SL to ring networks. The PoA of SL for general networks is smaller than $2(\ell + n)$, where $\ell$ stands for the length of the longest path a player may select and $n$ stands for the number of network nodes (Busch and Magdon-Ismail 2009). The worst case of SL discovered so far occurs in rings, where the PoA can be both as high as $n - 1$. The authors also conjectured that the matching upper bound on the PoA of SL in general networks equals the length of the longest cycle in the network minus 1, which equals both $n - 1$ and $\ell - 1$ for ring networks.

Because cooperation among autonomous players may be mutually beneficial even if the players selfishly try to optimize their own objectives, the PoA and PoS arguably conflate the effects of selfishness (which is still reserved in SE) and lack of coordination (which disappears in SE). For this reason, Andelman et al. (2009) refined the measures to be *$k$-Strong Price of Anarchy* ($k$-SPoA) and *$k$-Strong Price of Stability* ($k$-SPoS) with respect to the so called *$k$-Strong Equilibrium* ($k$-SE) that is a strategy profile in which no coalition of size at most $k$ has any joint deviation beneficial to all members. To be precise, the $k$-SPoA (resp. $k$-SPoS) is defined as the ratio of the social cost in the worst (resp. best) $k$-SE to that in a global optimum. The concept of $k$-SE generalizes both NE (which is a 1-SE) and SE (which is an $m$-SE in $m$-player games). Following convention, the “$k$-” is often omitted from the notions when integer $k$ equals the number of players in the game.

Employing $k$-SE has a potential to reduce the PoA, since every $k$-SE is a NE but not vice versa. In the extreme case of SE where $k$ equals the number of players, it has been shown that the SPoA is significantly lower than the PoA in many settings of competitive games (see, e.g., Albers 2009; Andelman et al. 2009). Nevertheless, SE mainly make sense in small networks where players have substantial information about the overall structures and can coordinate their actions (Jackson 2005). Emergent efforts have been devoted to investigating $k$-SE for
general $k$. It was shown that the $k$-SPoA falls in $\left[\frac{(l-1)(m-l+1)}{k-l+1}, \frac{2m(l-1)}{k-l+1}\right]$ for job scheduling game with $l(\leq k)$ unrelated machines and $m$ jobs (Andelman et al. 2009; Fiat et al. 2007), and falls in $[\max\{\frac{m}{k}, \sum_{i=1}^{m} \frac{1}{i}\}, \frac{m}{k} \sum_{i=1}^{k} \frac{1}{i}]$ for fair connection game with $m$ players (Albers 2009; Epsteina et al. 2009). Albeit these efforts, for small $k$ the existing results on $k$-SE are not so attractive in the aspect of practical applications. Even worse situations happen in some setting of the network creation game where no $k$-SE exists for any $k \geq 3$ (Andelman et al. 2009).

Concerning the major downside of the demanding SE, various works have studied its existence in particular families of games (Albers 2009; Andelman et al. 2009; Epsteina et al. 2009; Holzman and Law-yone 2003, etc.). Closely related to our SRLC model is the recent introduction of $\pi$-Lexicographic Improvement Property ($\pi$-LIP) by Harks et al. (2009). They showed that games of $\pi$-LIP always possess SE, and identified so called bottleneck congestion games, which enjoy the $\pi$-LIP and include job scheduling (Andelman et al. 2009), SL (Busch and Magdon-Ismail 2009) and hence SRLC (this paper) as special cases.

While our SRLC model, as well as $k$-SE, works with dynamic coalitions, other different approaches have been taken to study the effect of coalition formation in network games based on static coalitions which are formed according to an exogenous partition over the set of players (Cominetti et al. 2006; Harks 2009; Hayrapetyan et al. 2006). Under certain settings, the quality of the solution can deteriorate by an arbitrarily high factor in the presence of static coalitions.

In addition to the number of coalition members, other natural restrictions have been imposed to admissible coalitions in efforts to improve the quality of outcomes in the decentralized setting. Leonardi and Sankowski (2007) studied how the PoA and PoS of network formation games with Sharpley cost allocation are affected by allowing locally coordinated coalitions of players, where a group of players can form a coalition if they share a link. This kind of local cooperation does not necessarily lead to better PoA, and increases PoS from $\Theta(\log m)$ to $\Theta(m)$ in $m$-player games. Along a different line of network formation, Jackson and Wolinsky (2005; 1996) proposed pairwise stability, a variant of 2-SE, to investigate the (in)compatibility of overall societal welfare with incentives of self-interested individuals to form and sever network links. Some connection between pairwise stability and 2-SE was recently established by Calvó-Armengol and İlkiç (2009).

Our contributions Henceforth, we restrict our attention to SRLC, and will often omit the reference to SRLC in our presentation. We prove that when coordination within coalitions of size up to $3$ is allowed, the PoA of equilibrium outcomes drops from $n - 1$ (linear in the ring size $n$) to constant 2. More specifically, we establish $3$-SPoA $= 2$ against PoA $\geq 2$-SPoA $= n - 1$. Hence, using coordination in 2- and 3-player coalitions, we achieve a significant improvement in terms of the PoA, compared to noncooperative and pairwise coordination environments. Few selfish routing models in literature enjoy such an interesting property as SRLC: small-sized coalitions can greatly improve not only the gains of individual players but also the performance of the whole network.

We also show that the $m$-SPoA $\in [1 + \frac{2}{m}, 2]$ is strictly greater than 1, and the $k$-SPoA $\in [\frac{k-1}{k-2} - \varepsilon, 2]$ is at most 2, where $k \geq 3$ and $\varepsilon > 0$ can be arbitrarily small.

We strengthen the existence of SE (Harks et al. 2009) by showing that every optimal routing in SRLC is a SE. Since an optimal routing for SRLC is derivable in polynomial time (Wang 2005), an immediate algorithmic corollary says that a SE in SRLC can be found in $O(m \log m)$ time and in $O(m)$ time when $m \geq n$.

Paper organization In Section 2, we first define the SRLC model mathematically; then we show that every optimal routing of SRLC is a SE, yielding the existence of $k$-SE and $k$-SPoS =
1 for every \( k \). In Section 3, we obtain lower bounds of \( k \)-SPoA exhibited by concrete examples: \( \text{PoA} \geq 2 \)-SPoA \( \geq n - 1 \), \( k \)-SPoA \( \geq (k - 1)/(k - 2) \) for \( m > k \geq 3 \) and SPoA \( \geq 1 + 2/m \). In Section 4, we derive upper bounds on \( k \)-SPoA: \( n - 1 \) for \( k \leq 2 \), and 2 for all \( k \geq 3 \); in particular we establish \( 3 \)-PoA = 2 and complement the result with \( \text{PoA} = 2 \)-SPoA = \( n - 1 \). In Section 5, we conclude the paper with discussion on future work.

2 Model

Our model of Selfish Ring Load with Collusion (SRLC) is specified by triple \( \mathcal{I} = (R, (s_j, t_j)_{j=1}^{m}, k) \), which is called a SRLC instance or SRLC\(_k\) instance to emphasize that up to \( k \) players can form a coalition. As illustrated in Figure 1(a), the underlying network of \( \mathcal{I} \) is a ring \( R = (V, E) \) with node-set \( V = \{v_i \mid i = 1, 2, \ldots, n\} \) and link-set \( E = \{e_i = v_i v_{i+1} \mid i = 1, 2, \ldots, n\} \), where \( v_{n+1} = v_1 \). There are \( m \) players in \( \mathcal{I} \), named as 1, 2, \ldots, \( m \). Each player \( j \in [m] \equiv \{1, 2, \ldots, m\} \) corresponds to a source-destination node pairs \((s_j, t_j)\) in \( R \), where \( s_j \neq t_j \) is assumed to avoid triviality. The strategy-set of player \( j \in [m] \) consists of two internally disjoint \( s_j\)-\( t_j \) paths in \( R \), denoted by \( P_j \) and \( \bar{P}_j \), respectively. For convenience, let \( \bar{P}_j = P_j \) for every \( j \in [m] \).

A (feasible) routing \( \pi = \{Q_1, Q_2, \ldots, Q_m\} \) for the SRLC instance \( \mathcal{I} \) consists of \( m \) paths, where \( s_j\)-\( t_j \) path \( Q_j \in \{P_j, \bar{P}_j\} \) is the route (also called the \( \pi \)-route) selected for routing the one unit traffic requested by player \( j \in [m] \). In \( \pi \), each link \( e \in E \) bears a load

\[
\pi[e] \equiv |\{Q_j \mid e \in E(Q_j), j = 1, 2, \ldots, m\}|
\]

equal the number of routes in \( \{Q_1, Q_2, \ldots, Q_m\} \) each going through \( e \). The integrality of link loads will be used explicitly or implicitly in this paper. Correspondingly, a subgraph \( P \) of \( R \) (written as \( P \subseteq R \)) and its link set \( E(P) \) bear the maximum load

\[
\pi[P] \equiv \pi[E(P)] = \max\{\pi[e] \mid e \in E(P)\}.
\]

In a mild abuse of notation, we shall identify graph \( P \) with its link set \( E(P) \) when no confusion arises; particularly, we often abbreviate \( e \in E(P) \) to \( e \in P \). The maximum loads experienced by player \( j \in [m] \), and by the ring (the routing \( \pi \)) are

\[
\pi_j \equiv \pi[Q_j] \quad \text{and} \quad \pi_R \equiv \pi[R], \quad \text{respectively.}
\]

When referring to the maximum load of a player, a path, or the ring, we often omit the term “maximum” for short. To emphasize that a (maximum) load is experienced in routing \( \pi \), we often call this load as a \( \pi \)-load. A routing \( \pi^* \) for \( \mathcal{I} \) is optimal if \( \pi^*_R \) is minimum among all routings for \( \mathcal{I} \). The problem of finding \( \pi^* \) is a special case of the ring loading problem with integer demand splitting, which could be solved in a polynomial time (Wang 2005).

![Figure 1: The SRLC model and instance](image-url)
By a coalition we mean a set of players. In $\mathcal{I} = (R, (s_j, t_j)_{j=1}^m, k)$, only coalitions of at most $k$ players are allowed to form, where integer $k \in [m]$. Given a routing $\pi$ for $\mathcal{I}$, the deviation by a coalition $S$ (from $\pi$ refers to the change of the $\pi$-routes of all players in $S$). We say that $\pi$ is (resp. is not) resilient to the deviation by (coalition) $S$ if the deviation does not decrease the load of at least one player in $S$ (resp. decreases the loads of all players in $S$).Routing $\pi$ is called a $k$-Strong Equilibrium (k-SE) if $\pi$ is resilient to the deviation by any coalition of at most $k$ players. By definition, a $k$-SE is an $h$-SE for all $h \in [k]$, and 1-SE is exactly the classical Nash Equilibrium (NE). Therefore every $k$-SE $\pi = \{Q_1, Q_2, \ldots, Q_m\}$ of $\mathcal{I}$ is a NE, and it is resilient to the deviation by coalition $\{j\}$ of a single player, satisfying the following NE inequalities:

$$\pi[Q_j] \geq \pi[Q_j] - 1 = \pi_j - 1 \text{ for all } j = 1, 2, \ldots, m.$$ (2.1)

An $m$-SE is identical with the classical Strong Equilibrium (SE). Since SRLC falls within the general framework of the bottleneck congestion game, it possesses so called $\pi$-Lexicographical Improvement Property, which implies the existence of SE (Harks et al. 2009); so every SRLC instance admits at least one k-SE. Conversely, the strong stability of optimal routings established below strengthens this existence.

**Theorem 2.1.** Every optimal routing in SRLC is a SE.

*Proof.* Suppose on the contrary that there exists an optimal routing $\pi^* = \{Q_1, Q_2, \ldots, Q_m\}$ of some SRLC instance $\mathcal{I}$ that is not resilient to deviation by coalition $[h]$ for some $h \in [m]$. Thus for routing $\pi = \{Q_1, Q_2, \ldots, Q_h, Q_{h+1}, \ldots, Q_m\}$ of $\mathcal{I}$, we have $\pi_j < \pi_j^* \leq \pi_j^*$ for every player $j \in [h]$ in the coalition. Since $\pi_R \geq \pi_R^*$, there exists link $e \in E \setminus \cup_{j \in [h]} E(Q_j) \neq \emptyset$ such that $\pi[e] = \pi_R \geq \pi_R^* \geq \pi^*\{e\}$. However, we deduce from $e \in \cap_{j \in [h]} Q_j$ that $\pi[e] = \pi^*[e] - h \leq \pi[e]^* - 1$, a contradiction.

In competitive games where SE exists and an initial solution could be imposed centrally, SE is usually a preferable candidate for the initial setting. Nevertheless, computing SE is in general prohibitive due to its NP-completeness. In contrast, given a SRLC instance, its optimal routing is derivable by Wang’s (2005) algorithm for the ring loading problem with integer demand splitting. The algorithm in combination with Theorem 2.1 implies the following polynomial time solvability of computing SE in SRLC.

**Corollary 2.2.** A SE in SRLC can be found in $O(m \log m)$ time and in $O(m)$ time when $m \geq n$.

Let $\pi^*$ be an optimal routing for SRLC$_k$ instance $\mathcal{I}$. The $k$-Strong Price of Anarchy (k-SPoA) of $\mathcal{I}$ is defined as the minimum value $\beta$ such that $\pi_R^*/\pi_R^* \leq \beta$ holds for every k-SE $\pi$ of $\mathcal{I}$. The $k$-Strong Price of Stability (k-SPoS) of $\mathcal{I}$ is defined as the minimum value $\beta$ such that $\pi_R^*/\pi_R^* \leq \beta$ holds for some k-SE $\pi$ of $\mathcal{I}$. The notion of the k-SPoA (resp. k-SPoS) extends to the SRLC$_k$ problem of all SRLC$_k$ instances, whose k-SPoA (resp. k-SPoS) is set to be the supremum of k-SPoA (resp. k-SPoS) over all SRLC$_k$ instances. Note that 1-SPoA is the same as the standard PoA (Koutsoupias and Papadimitriou 1999), and m-SPoA is often abbreviated to SPoA. Clearly,

$$1 \leq \text{PoS} = 1-\text{SPoS} \leq 2-\text{SPoS} \leq \cdots \leq k-\text{SPoS} \leq k-\text{SPoA} \leq \cdots \leq 2-\text{SPoA} \leq 1-\text{SPoA} = \text{PoA}.$$ (2.2)

The string of inequalities and Theorem 2.1 give the following immediate corollary.

**Corollary 2.3.** For any integer $k \in [m]$, every SRLC$_k$ instance admits at least one k-SE, and PoS = k-SPoS = 1.
3 Lower Bounds on Price of Anarchy of SRLC

In this section, we will establish several lower bounds on k-SPoA of SRLC via concrete SRLC_k instances. Let us begin with an example from Busch and Magdon-Ismail (2009).

**SRLC_2 instance** \( I_2 = (R, (s_j, t_j)_{j=1}^n, 2) \). There are \( m = n \geq 3 \) players. Ring \( R = (V, E) \) with \( V = \{v_i \mid i = 1, 2, \ldots, n\} \) has size \( n \). Pairs of consecutive nodes are source-destination pairs \( \{s_j, t_j\} = \{v_{j}, v_{j+1}\}, j = 1, 2, \ldots, n \), where \( v_{n+1} \equiv v_1 \).

It is easy to check that the unique optimal routing \( \pi^* \) for \( I_2 \) with \( \pi^*_R = 1 \) consists of \( n \) routes each of one link, and a 2-SE \( \pi \) for \( I_2 \) with \( \pi_R = n - 1 \) consists of \( n \) routes each of \( n - 1 \) links. (See Figure 1(b, c) for an illustration with \( n = 4 \).) Hence, using NE inequalities (2.2), we obtain the following observation.

**Lemma 3.1.** PoA \( \geq 2 \text{-} \text{SPoA} \geq \frac{\pi_R}{\pi^*_R} = n - 1 \) for SRLC_2 instance \( I_2 \).

For the setting of \( I_2 \), it is not hard to see that permitting coalitions of sizes at most 3 guarantees that 3-NE must be the unique optimal routing. To study k-SPoA of SRLC_k for \( k \geq 3 \), we need more complicated settings.

**SRLC_k instance** \( I_{k, \ell} = (R, (s_j, t_j)_{j=1}^m, k) \). There are \( m = n \equiv (2k - 3)\ell - 1 \) players, where \( k \) and \( \ell \) are integers satisfying \( k \geq 3 \) and \( \ell \geq 2 \). Ring \( R = (V, E) \) with \( V = \{v_i \mid i = 1, 2, \ldots, n\} \) has size \( n \). The source and destination of player \( j \in [m] \) are \( s_j = v_j \) and \( t_j = v_{j+(k-2)\ell} \) (mod \( n \)), where \( v_0 \equiv v_n \). (See Figure 2 for an illustration.)

Figure 2: Two routings for SRLC_k instance \( I_{k, \ell} \) with \( k = 3 \) and \( \ell = 2 \), where \( n = m = 5 \)

**Theorem 3.1.** k-SPoA \( \geq \frac{k-1}{k-2} - \frac{1}{(k-2)\ell} \) for SRLC_k instance \( I_{k, \ell} \) with \( k \geq 3 \) and \( \ell \geq 2 \).

**Proof.** Let \( \pi \) and \( \pi' \) denote the two routings for \( I_{k, \ell} \) in which every player \( j \in [m] \) adopts longer route \( Q_j \) with \( |E(Q_j)| = (k-1)\ell - 1 \) and shorter route \( \bar{Q}_j \) with \( |E(\bar{Q}_j)| = (k-2)\ell \), respectively. It is easy to see that

\[
\pi[e] = (k-1)\ell - 1 \quad \text{and} \quad \pi'[e] = (k-2)\ell \quad \text{for all } e \in E,
\]

yielding \( \frac{\pi_R}{\pi^*_R} = \frac{k-1}{k-2} - \frac{1}{(k-2)\ell} \). It remains to show that \( \pi \) is a \( k \)-SE for \( I_{k, \ell} \).

Suppose otherwise. Then there exists coalition \( S \subseteq [m] \) with \( |S| \leq k \) such that \( \pi \) is not resilient to the deviation by \( S \). Let \( \pi'' \) be the routing obtained from \( \pi \) via the deviation by \( S \). Then for any fixed player \( s \in S \), its \( \pi'' \)-load is lower than its \( \pi \)-load, saying

\[
\pi''_s \equiv \pi''[Q_s] \leq \pi_s - 1 = (k-1)\ell - 2 \Rightarrow \pi''_R \leq \pi_R = (k-1)\ell - 1.
\] (3.1)
For any \( e \in E \), let \( \lambda_e \equiv |\{ j | e \in Q_j, j \in S \}| \) (resp. \( \bar{\lambda}_e \equiv |\{ j | e \in \bar{Q}_j, j \in S \}| \) denote the number of players in \( S \) whose \( \pi \)-routes (resp. \( \pi'' \)-routes) contain \( e \). Then

\[
\lambda_e + \bar{\lambda}_e = |S| \quad \text{and} \quad \pi''[e] = \pi[e] - \lambda_e + \bar{\lambda}_e = (k-1)\ell - 1 - \lambda_e + \bar{\lambda}_e \quad \text{for all} \ e \in E; \tag{3.2}
\]

\[
\sum_{e \in E} \bar{\lambda}_e = \sum_{j \in S} |E(Q_j)| = |S|(k-2)\ell. \tag{3.3}
\]

The (in)equalities in (3.1) and (3.2) assure the following upper bounds on \( \bar{\lambda}_e \) for all \( e \in E = E(Q_s \cup \bar{Q}_s) \):

\[
\bar{\lambda}_e \leq \left\lfloor \frac{|S|}{2} \right\rfloor \quad \text{for all} \ e \in Q_s, \quad \text{and} \quad \bar{\lambda}_e \leq \left\lfloor \frac{|S| - 1}{2} \right\rfloor \quad \text{for all} \ e \in \bar{Q}_s, \tag{3.4}
\]

which would imply a contradiction to (3.3). Indeed, if \( \bar{\lambda}_e \geq \left\lfloor \frac{|S|}{2} \right\rfloor + 1 \) for some link \( e \in Q_s \), then \( \bar{\lambda}_e > \left\lfloor \frac{|S|}{2} \right\rfloor - 1 \geq |S| - \bar{\lambda}_e \); it follows from (3.2) that \( \bar{\lambda}_e > \lambda_e \) and \( \pi'' \geq \pi''[e] > (k-1)\ell - 1 \), a contradiction to (3.1). If \( \bar{\lambda}_e \geq \left\lfloor \frac{|S| - 1}{2} \right\rfloor + 1 \) for some link \( e \in \bar{Q}_s \), then it follows that \( \bar{\lambda}_e \geq \left\lfloor \frac{|S| + 1}{2} \right\rfloor - 1 \geq |S| - \bar{\lambda}_e \); by (3.2) we obtain \( \bar{\lambda}_e \geq \lambda_e \) and \( \pi'' \geq \pi''[e] \geq (k-1)\ell - 1 \) contradicting (3.1).

In the following we will see that no matter whether \( |S| \) is odd or even, the inequalities in (3.4) give a violation \( \sum_{e \in E} \bar{\lambda}_e < |S|(k-2)\ell \) of (3.3).

**Case 1**: \( |S| \) is odd. We have \( \bar{\lambda}_e \leq \left\lfloor \frac{|S| - 1}{2} \right\rfloor \) for all \( e \in E \), and the violation as

\[
\sum_{e \in E} \bar{\lambda}_e \leq n \left\lfloor \frac{|S| - 1}{2} \right\rfloor = ((2k-3)\ell - 1) \left\lfloor \frac{|S| - 1}{2} \right\rfloor \leq |S|(k-2)\ell.
\]

**Case 2**: \( |S| \) is even. We have \( \bar{\lambda}_e \leq \left\lfloor \frac{|S| - 1}{2} \right\rfloor = \left\lfloor \frac{|S|}{2} \right\rfloor - 1 \) for all links \( e \in \bar{Q}_s \) and \( \bar{\lambda}_e \leq \left\lfloor \frac{|S|}{2} \right\rfloor = \frac{|S|}{2} \) for all \( e \in Q_s \), yielding

\[
\sum_{e \in E} \bar{\lambda}_e = \sum_{e \in Q_s} \bar{\lambda}_e + \sum_{e \in \bar{Q}_s} \bar{\lambda}_e \\
\leq |E(\bar{Q}_s)| \left( \frac{|S|}{2} - 1 \right) + |E(Q_s)| \cdot \left\lfloor \frac{|S|}{2} \right\rfloor \\
= (k-2)\ell \left( \frac{|S|}{2} - 1 \right) + ((k-1)\ell - 1) \left\lfloor \frac{|S|}{2} \right\rfloor \\
= |S|(k-2)\ell + \frac{|S|\ell - 1}{2} - (k-2)\ell.
\]

The violation is implied by \( \frac{|S|\ell - 1}{2} - (k-2)\ell < 0 \), which is equivalent to \( 4 - \left\lfloor \frac{|S|}{2} \right\rfloor < 2k - |S| \) as \( \ell \geq 2 \). It remains to show \( 2k - |S| \geq 4 \). This is clearly true because \( k \geq 3 \), \( k \geq |S| \), and \( |S| \) is an even number. \( \square \)

Theorem 3.1 will be used in Section 4 to establish lower bound 2 on the 3-SPoA of SRLC. Moreover, from the theorem we see that for every \( k \geq 3 \) there is an instance \( T_{k,\ell} \) with \( k \)-SPoA > 1. Note that in \( T_{k,\ell} \) the largest coalition size \( k \) is at most 3/5 of the number of players \( m \). Can one expect that \( k \)-SPoA approaches 1 as \( k \) turns to \( m \)? The answer is negative, as shown by the following SRLC in instance whose SPoA > 1 for the extreme case of \( k = m \).

**SRLC** instance \( J_m = (R, (s_j, t_j)_{j=1}^m, m) \). There are an even number \( m \geq 4 \) of players. Ring \( R \) has size \( n = 2m - 2 \). Player 1 has source \( s_1 = v_1 \) and destination \( t_1 = v_2 \); player \( j \in [m] \setminus \{1\} \) has source \( s_j = v(j-2)(m-2)+1 \) (mod \( n \)) and destination \( t_j = v(j-2)(m-2)+m \) (mod \( n \)). (See Figure 3 for an illustration.)

In our discussion on SRLC instance \( J_m \), we suppose that nodes \( v_1, v_2, \ldots, v_n \) are encountered in this order when traversing \( R \) in a clockwise direction, and that \( Q_s \) is the route from \( s_j \) to \( t_j \) routed clockwise for \( j = 1, 2, \ldots, m \). Observe that player 1 is very special in that its
route can be very “short” $Q_1$ of one link or very “long” $\bar{Q}_1$ of $2m-3$ link, while the both paths of other players are always of $m-1$ links (half length of the ring). As the following lemma shows, depending on whether player 1 chooses its shorter path or longer path, any other player experiences maximum load at least $m/2$ or at least $m/2 + 1$.

**Lemma 3.2.** Let $\pi''$ be any routing for $J_m$. Then $\pi''_j \geq \frac{m}{2}$ for every $j \in \{m\} \setminus \{1\}$, and $\pi''_j \geq \frac{m}{2} + 1$ for every $j \in \{m\} \setminus \{1\}$ if $\bar{Q}_1 \in \pi''$.

*Proof.* Suppose $\pi'' = \{P_1, P_2, \ldots, P_m\}$. For any $j \in \{m\} \setminus \{1\}$, let $e$ and $f$ denote the two end links of $P_j$. For all $l \in \{m\} \setminus \{1, j\}$, observe that $|E(P_j)| = m - 1 = |E(P_l)|$. Since $|E| = n = 2(m-1)$ and $\{s_j, t_j\} \neq \{s_l, t_l\}$, we see that both $P_j \cap P_l$ and $P_l \cap Q_1$ contain either $e$ or $f$. So the $\pi''$-routes of at least a half of players in $\{m\} \setminus \{1, j\}$ (resp. $\{m\} \setminus \{j\}$ when $\bar{Q}_1 \in \pi''$) contain $e$ or $f$, say $e$, implying

$$\pi''[e] \geq \left\lceil \frac{|\{m\} \setminus \{1, j\}|}{2} \right\rceil + |\{e\} \cap P_j|, \text{ and } \pi''[e] \geq \left\lceil \frac{|\{m\} \setminus \{1\}|}{2} \right\rceil + |\{e\} \cap P_j| \text{ when } \bar{Q}_1 \in \pi''.$$ 

Since $e \in P_j$ and $m$ is an even number, we see that $\pi''_j \geq \pi''[e]$ is at least $\frac{m-2}{2} + 1 = \frac{m}{2}$, and is at least $\lceil \frac{m-1}{2} \rceil + 1 = \frac{m}{2} + 1$ when $\bar{Q}_1 \in \pi''$. 

As depicted in Figure 3, $\pi = \{Q_1, Q_2, \ldots, Q_m\}$ and $\pi' = \{Q_1, \bar{Q}_2, \ldots, \bar{Q}_m\}$ are two routings of $J_m$ in which only player 1 chooses the same route, $Q_1$ of single link $e_1$. It is easy to verify that $\pi_R = \pi[\{e_1\}] = \frac{m}{2} + 1$ and $\pi[E \setminus \{e_1\}] = \frac{m}{2} = \pi'_R$.

**Theorem 3.2.** $SPoA \geq \frac{\pi_R}{\pi'R} = 1 + \frac{2}{m}$ for $SRLC_m$ instance $J_m$ with even $m \geq 4$.

*Proof.* It suffices to show that $\pi$ is an $m$-SE for $J_m$. Suppose on the contrary that $\pi$ is not resilient to the deviation by some coalition $S \subseteq \{m\}$. Let $\pi''$ be the routing obtained from $\pi$ via the deviation by $S$. Then NE inequalities (2.1) read $\pi''_j \leq \pi_j - 1 \leq \pi'_R - 1 = \frac{m}{2}$ for each $j \in S$. It follows from Lemma 3.2 that $\bar{Q}_1 \notin \pi''$, i.e., $1 \notin S$, and the inequalities hold with equalities throughout, giving

$$\pi'_j = \frac{m}{2} \text{ and } \pi_j = \frac{m}{2} + 1 \text{ for each } j \in S. \quad (3.5)$$

Thus $\pi[E \setminus \{e_1\}] = \frac{m}{2}$ enforces

$$e_1 \in Q_j, \text{ and } 1 \notin S \Rightarrow |Q_j| = m - 1 \Rightarrow e_m = v_mv_{m+1} \in Q_j \text{ for each } j \in S. \quad (3.6)$$

Let $U$ denote the set of players whose source is $v_i$ for some odd $i \in [3, m - 1]$. Then $U \cap S = \emptyset$ because $e_1 \notin Q_j \ni e_m$ for every $j \in U$. It follows that from $\pi$ to $\pi''$ all players in $U$ remains unchanged, and their routes $Q_j (j \in U)$ in both $\pi$ and $\pi''$ all contain $e_m$. In addition to players in $U$, by (3.6) all players $j \in S$ have their $\pi''$-routes $\bar{Q}_j$ go through $e_m$. So
\[
\pi''[e_m] \geq |U| + |S| = \frac{m}{2} - 1 + |S|.
\]

On the other hand, \(\pi''[e_m]\) does not exceed the \(\pi''\)-load of any player in \(S\), which is at most \(\frac{m}{2}\) by (3.5). Therefore we have \(\frac{m}{2} \geq \frac{m}{2} - 1 + |S|\), which enforces \(|S| = 1\). Let \(j\) be the player consisting of \(S\). It is straightforward to check that \(\pi''_j \geq \pi[Q_j] + 1 = \frac{m}{2} + 1\), a contradiction to \(\pi''_j = \frac{m}{2}\) in (3.5).

For odd integer \(m = 2h + 1 \geq 7\), we construct SRLC \(m\) instance \(J_m = J_{2h+1}\) with SPoA \(\geq 1 + \frac{1}{m-1}\) based on \(J_{2h}\) as follows: introduce player \(2h + 1\) to \(J_{2h}\), and set \(s_{2h+1} = v_3, t_{2h+1} = v_4\). Define route \(Q_{2h+1} = v_3v_4\) for player \(2h + 1\) and routes \(Q_1, \ldots, Q_{2h}\) for players \(1, \ldots, 2h\) as in \(J_{2h}\). Then \(\{Q_1, \ldots, Q_m\}\) is an \(m\)-SE for \(J_m\) with load \(\frac{m+1}{2}\), and \(\{Q_1, Q_2, \ldots, Q_{m-1}, Q_m\}\) is an optimal routing for \(J_m\) with load \(\frac{m-1}{2}\). The ratio between these two loads equals \(1 + \frac{2}{m-1}\).

4 Upper Bounds on Price of Anarchy of SRLC

The upper bounds to be established in this section imply that the 2-SPoA and 3-SPoA of the SRLC problem are \(n - 1\) and 2, respectively.

4.1 Upper bounds

Recall from Lemma 3.1 that the both PoA and 2-SPoA of SRLC is at least \(n - 1\). A general bound in Busch and Magdon-Ismail (2009) implies PoA \(\leq 2(n - 1 + \log n)\). The \(O(n)\) additive gap between the upper and lower bounds is bridged by the following theorem.

Theorem 4.1. For the SRLC2 problem, PoA = 2-SPoA = \(n - 1\).

Proof. By Lemma 3.1, it suffices to show PoA \(\leq n - 1\). Suppose on the contrary that some SRLC instance \(I = (R, (s_j, t_j)_{j=1}^n, 1)\) has an optimal routing \(\pi^*\) with load \(\pi_R^* = \Lambda^*\), and a NE routing \(\pi = \{Q_1, Q_2, \ldots, Q_m\}\) with load \(\pi_R^* > (n - 1)\Lambda^*\). So there is a link \(f \in E\) of load \(\pi(f) \geq (n - 1)\Lambda^* + 1\), where we recall that link load \(\pi(f)\) is an integer. Assume without loss of generality that \(f \in Q_j\) for all \(j \in [(n - 1)\Lambda^* + 1]\), and \(Q_j \in \pi^*\) for all \(j \in [(n - 2)\Lambda^* + 1]\) (as \(\pi^*[f] \leq \Lambda^*\)).

If there is a link \(g \in E \setminus \{f\}\) such that \(\pi^*[g] < (n - 1)\Lambda^*\), then it follows from NE inequalities (2.1) that \(E(Q_j) \setminus \{f, g\} = E(Q_j) \setminus \{g\} \neq \emptyset\) for all \(j \in [(n - 2)\Lambda^* + 1]\), implying that there must exist link \(d \in \cup_{j=1}^{(n-2)\Lambda^*+1} E(Q_j) \setminus \{f, g\}\) whose load \(\pi^*[d]\) is at least the average link load on \(\cup_{j=1}^{(n-2)\Lambda^*+1} E(Q_j) \setminus \{f, g\}\) in the optimal routing \(\pi^*\), that is

\[
\pi^*[d] \geq \frac{\sum_{j=1}^{(n-2)\Lambda^*+1} |E(Q_j) \setminus \{f, g\}|}{\sum_{j=1}^{(n-2)\Lambda^*+1} |E(Q_j) \setminus \{f, g\}|} \geq \frac{(n-2)\Lambda^*+1}{|E(Q_j) \setminus \{f, g\}|} \geq (n-2)\Lambda^*+1 = \Lambda^* + \frac{1}{n-2}.
\]

This contradicts \(\pi^*[d] \leq \pi^*_R = \Lambda^*\), and thus shows that \(\pi^*[g] \geq (n - 1)\Lambda^*\) for all \(g \in E \setminus \{f\}\). Hence \(\sum_{e \in E} \pi^*[e] = \pi^*[f] + \sum_{g \in E \setminus \{f\}} \pi^*[g]\) is lower bounded by \((n-1)\Lambda^* + 1 + (n-1)(n-2)\Lambda^* > n(n-1)\Lambda^*\).

On the other hand, as \(|E(Q_j)| \leq n - 1\) for all \(j \in [m]\), we see that \(\sum_{e \in E} \pi^*[e] = \sum_{j \in [m]} |E(Q_j)|\) is upper bounded by \(m(n-1)\). So from \(n(n-1)\Lambda^* \leq m(n-1)\) we derive \(m > n\Lambda^* \geq \sum_{e \in E} \pi^*[e]\). However, since the \(\pi^*\)-route of each player contributes at least 1 to the total load \(\sum_{e \in E} \pi^*[e]\) on the ring, we have \(\sum_{e \in E} \pi^*[e] \geq m\), a contradiction.

Notice that Theorem 4.1 excludes the possibility of obtaining constant upper bounds on PoA and 2-SPoA of SRLC. In contrast, upper bound 2 will be established for 3-SPoA.

Theorem 4.2. For the SRLCk problem with \(3 \leq k \leq m\), k-SPoA \(\leq 2\).
The technical proof of Theorem 4.2 is left to the next subsection. Now we present an immediate corollary of Theorems 3.1 and 4.2, which implies exact value 2 of 3-SPoA for SRLC.

**Theorem 4.3.** For the SRLC\(_k\) problem with \(k \geq 3\), k-SPoA \(\in [\frac{k-1}{k-2}, 2]\). In particular, for the SRLC\(_3\) problem, 3-SPoA = 2.

**Proof.** Letting \(\ell \to \infty\) in Theorem 3.1, we get SRLC\(_k\) instance \(\mathcal{I}_{3, \ell}\) whose k-SPoA \(\geq \frac{k-1}{k-2} - \frac{1}{(k-2)\ell} = \frac{k-1}{k-2} - \frac{1}{\ell} \to \frac{k-1}{k-2}\). Hence the k-SPoA of the SRLC\(_k\) problem is at least \(\frac{k-1}{k-2}\). The conclusion is therefore instant from Theorem 4.2. \(\square\)

### 4.2 Proof of Theorem 4.2

The whole subsection is devoted to proving Theorem 4.2. In view of the string of inequalities (2.2), to prove the theorem, we only need to show that 3-SPoA \(\leq 2\). By contradiction, we suppose on the contrary: there exists a SRLC\(_3\) instance \(\mathcal{I} = (R, (s_j, t_j)_{j=1}^m, 3)\) satisfying the following.

**Assumption 1.** Instance \(\mathcal{I}\) admits a 3-NE \(\pi = \{Q_1, \ldots, Q_\ell, Q_{\ell+1}, \ldots, Q_m\}\) and an optimal routing \(\pi^* = \{\bar{Q}_1, \ldots, \bar{Q}_\ell, \bar{Q}_{\ell+1}, \ldots, \bar{Q}_m\}\) such that \(\Lambda^* = \pi_R^*\) and \(\pi[e] = \pi_R \equiv \Lambda \geq 2\Lambda^* + 1\) for some \(e \in E\).

Given any coalition \(S\) of players, by \(\pi^S\) we mean the routing for \(\mathcal{I}\) obtained from \(\pi\) by changing \(Q_j\) to \(\bar{Q}_j\) for all \(j \in S\). By Assumption 1, \(\pi\) must be resilient to the deviation by coalition \(S\) of at most three players. This is equivalent to the following.

**Proposition 2.** For any coalition \(S \subseteq [m]\) with \(|S| \leq 3\), there exists at least one player \(j \in S\) such that \(\pi_j \leq \pi_j^S\). \(\square\)

Our approach is to derive a coalition \(S\) contradicting to Proposition 2. The contradiction to the stability of \(\pi\) will establish Theorem 4.2. As usual, for any subgraphs \(G_1\) and \(G_2\) of \(R\), we use \(G_1 \cup G_2\) (resp. \(G_2 \cap G_1\)) to denote the subgraph of \(R\) with node set \(V(G_1) \cup V(G_2)\) (resp. \(V(G_1) \cap V(G_2)\)) and link set \(E(G_1) \cup E(G_2)\) (resp. \(E(G_1) \cap E(G_2)\)).

Observe from Assumption 1 that players who adopt different routes in \(\pi\) and \(\pi^*\) are \(1, \ldots, \ell\). They play an important role in our proof, because they give us a way to compare \(\pi\) and \(\pi^*\). We call them switching players. The players of coalition \(S\), which would deviate and thus contradict Proposition 2, will be selected from the set \([\ell]\) of switching players.

We briefly sketch the idea behind our arguments. (More detailed intuitions are given in the corresponding subsections to facilitate understanding on technical propositions and claims over there.) Intuitively, switching players of high \(\pi\)-loads and their \(\pi\)-routes jointly cover the whole ring \(R\) are more likely to form the deviating coalition \(S\) as desired.

Due to the large difference \(\pi_R - \pi_R^* \geq \Lambda^* + 1\), there must exist a large number \(\ell\) of switching players (cf. (4.1)) who go through all of most congested links in \(\pi\) (cf. (4.2)). Furthermore using (contradiction deduced from) unilateral deviation, we obtain a pair of switching players whose \(\pi\)-routes jointly cover the ring \(R\) (see Claim 4).

In order to “assure” this pair of players of high incentives to deviate, among all candidates, we choose a pair such that their \(\pi\)-routes are as long as possible. Suppose that switching players 1 and \(\ell\) form such a pair (cf. (4.3) and (4.4)). Using bilateral deviation of 1 and \(\ell\), we obtain a most congested link \(e \in \bar{Q}_\ell \subseteq Q_1\) with \(\pi[e] = \Lambda \geq 2\Lambda^* + 1\) and (second) most congested link \(e' \in Q_1 \subseteq \bar{Q}_\ell\) with \(\pi[e'] \geq \Lambda - 1\) (cf. Claim 7 and Figure 4(a)). In particular, player 1 experiences the maximum \(\pi\)-load \(\Lambda\) at \(e\), and is considered as a bottleneck player.
In addition to the highly congested links \( e \) and \( e' \), we find more links of high loads (at least \( \Lambda - 2 \)) in \( \pi \) which are contained in \( Q_1 \cup Q_\ell \) (cf. Claim 8 and (4.8)). These links (whose set is written as \( F \)) help us to identify players in coalition \( S \) in an efficient way: given any three links in \( F \) one of which has \( \pi \)-load at least \( \Lambda - 1 \), they are all used in \( \pi \) and avoided in \( \pi^* \) by some switching player (cf. Proposition 11).

By this property of \( F \), the highly congested links \( e, e' \) of \( \pi \)-loads \( \geq \Lambda - 1 \), and the choices switching players \( 1, \ell \) (their \( \pi \)-routes covering \( R \) as long as possible), we discover the other two heavily loaded switching players 2 and 3 such that their \( \pi \)-routes both contain all \( F \)-links on \( Q_1 \), and jointly contain all links on \( Q_\ell \) of loads at least \( \Lambda - 1 \) (cf. Figure 5(a), Claims 14 and 17). In particular, ring \( R \) are covered by \( \pi \)-routes of players 1, 2, and 3 together.

Subsequently, we show that players 2 and 3 are also bottlenecks with maximum \( \pi \)-load (cf. Claims 18 and 19). Finally, the bottleneck players 1, 2 and 3 would form the desired coalition \( S \) showing a trilateral deviation and a contradiction to Proposition 2.

Sections 4.2.1, 4.2.2, 4.2.3 below are devoted to discussing unilateral, bilateral and trilateral deviations, respectively.

### 4.2.1 Unilateral deviations

For any most congested link \( e \) with \( \pi[e] = \pi_R \geq 2\Lambda^* + 1 \) as in Assumption 1, it is clear that

\[
\ell \geq \pi[e] - \pi^*[e] \geq 2\Lambda^* + 1 - \Lambda^* = \Lambda^* + 1 > \pi^*[\cap_{j \in [\ell]} \bar{Q}_j]
\]

\[
\Rightarrow \cap_{j \in [\ell]} E(\bar{Q}_j) = \emptyset \Rightarrow e \in \cup_{j \in [\ell]} Q_j
\]

Hence there are a “large” number \( \ell \geq \Lambda^* + 1 \) of switching players, and their \( \pi \)-routes jointly cover all of most congested links in \( \pi \). For the 3-SE \( \pi \), NE inequalities (2.1) restated below correspond to the case of \( |S| = 1 \) in Proposition 2.

**Proposition 3.** \( \pi[\bar{Q}_j] \geq \pi_j - 1 = \pi[Q_j] - 1 \) for all \( j \in [m] \).

Further to the coverage of \( \cap_{j \in [\ell]} Q_j \) over all of most congested links, using unilateral deviation, i.e., Proposition 3, we show that the whole ring is covered by \( \pi \)-routes of some pair of switching players.

**Claim 4.** There exist switching players \( p, q \in [\ell] \) such that \( Q_p \cup Q_q = R \). i.e., \( E(\bar{Q}_p \cap \bar{Q}_q) = \emptyset \).

**Proof.** Let \( e \) with \( \pi[e] = \pi_R \geq 2\Lambda^* + 1 \) be the most congested link as in Assumption 1. Since \( e \in \cup_{j \in [\ell]} Q_j \) by (4.2), we assume without loss of generality that \( e \in Q_1 \). It follows from Proposition 3 that there exists link \( e' \in Q_1 \) with \( \pi[e'] = \pi[Q_1] \geq \pi[Q_j] - 1 \geq \pi[e] - 1 \geq 2\Lambda^* \).

Let \( J \equiv \{ j \in [\ell] \mid \{e, e'\} \subseteq Q_j \subseteq [\ell] \setminus \{1\} \} \) denote the set of switching players whose \( \pi \)-routes contain both links \( e, e' \) (or equivalently whose \( \pi^* \)-routes avoid both \( e, e' \)). Examining the contributions of players’ routes to \( \pi[e] + \pi[e'] \) and that to \( \pi^*[e] + \pi^*[e'] \), we see that each player in \( J \) contributes 2 to \( \pi[e] + \pi[e'] \) and 0 to \( \pi^*[e] + \pi^*[e'] \), while each player outside \( J \) contributes to \( \pi[e] + \pi[e'] \) at most as much as what it contributes to \( \pi^*[e] + \pi^*[e'] \). Therefore

\[
2|J| \geq (\pi[e] + \pi[e']) - (\pi^*[e] + \pi^*[e']) \geq (4\Lambda^* + 1) - (2\Lambda^*) = 2\Lambda^* + 1 \Rightarrow |J| \geq \Lambda^* + 1,
\]

where we use \( \pi[e] \geq 2\Lambda^* + 1, \pi[e'] \geq 2\Lambda^* \) and \( \pi^*[e], \pi^*[e'] \leq \pi_R^* = \Lambda^* \). Since the \( \pi \)-routes of all players in \( J \) meet at \( e \) and \( e' \), the “large” size \( \geq \Lambda^* + 1 \) of \( J \) implies the coverage of the ring \( R \) by \( \pi \)-routes two players in \( \{1\} \cup J \), proving the claim.

Suppose otherwise: \( Q_j \cup Q_l \neq R \) for every \( j, l \in \{1\} \cup J \). Then \( Q_1 \nsubseteq Q_j \) for every \( j \in J \). By the positions of \( e \) and \( e' \) on \( R \) (cf. Figure 4(a)), one of end links of \( Q_1 \), written as \( f \), is disjoint from \( Q_j \), i.e., \( f \notin Q_j \), for all \( j \in J \). However, since \( J \subseteq [\ell] \), it follows that \( Q_j \in \pi^* \) for all \( j \in J \), and \( \pi^*[f] \geq |J| \geq \Lambda^* + 1 \), which is absurd as \( \pi^*[f] \leq \Lambda^* \).
As a corollary of the above proof, we see that for any pair of links \( f, g \in E \) of total \( \pi \)-load \( \pi[f] + \pi[g] \geq 4\Lambda^* + 1 \), the set \( \{ j \in [\ell] \mid \{ e, f \} \subseteq Q_j \} \) contains \( \Lambda^* + 1 \) switching players whose \( \pi \)-routes all go through \( f \) and \( g \).

4.2.2 Bilateral deviations

By convention \( E' \triangle E'' = (E' \setminus E'') \cup (E'' \setminus E') \) denotes the symmetric difference of sets \( E' \) and \( E'' \). For any subgraphs \( G_1 \) and \( G_2 \) of \( R \), by \( G_1 \triangle G_2 \) we mean the graph spanned by links in \( E(G_1) \triangle E(G_2) \); by \( G_1 \setminus E(G_2) \) we mean the graph obtained \( G_1 \) by removing \( E(G_1) \cap E(G_2) \). We assume \( \pi[\emptyset] = 0 \), meaning that the empty set \( \emptyset \) experiences no load. Similarly, if a subgraph \( G \) of \( R \) has no link, then \( \pi[G] = 0 \).

The following technical result will be repeatedly used in our discussion on bilateral deviation: its proof shows a violation of Proposition 2 with \( |S| = 2 \).

**Proposition 5.** For any pair of distinct players \( p, q \in [m] \), either \( \pi[Q_p \cap Q_q] \geq \Lambda - 2 \), or \( \pi[Q_p \triangle Q_q] = \Lambda \).

**Proof.** Suppose that \( \pi[Q_p \cap Q_q] \leq \Lambda - 3 \) for some distinct \( p, q \in [m] \). Considering the routing \( \pi' = \pi^{[p,q]} \), we have

\[
\pi'[f] = \pi[f] \quad \text{for all } f \in Q_p \triangle Q_q, \quad \text{and} \quad \pi'[f] = \pi[f] + 2 \quad \text{for all } f \in Q_p \cap Q_q.
\]

If \( \pi[Q_p \triangle Q_q] \leq \Lambda - 1 \), then \( \pi'[Q_p \triangle Q_q] \leq \Lambda - 1 \), and the most congested link, with \( \pi \)-load \( \pi_R = \Lambda \), is contained by \( Q_p \cap Q_q \), enforcing \( \pi_p = \pi_q = \Lambda \). Moreover, from \( \pi[Q_p \cap Q_q] \leq \Lambda - 3 \) it can be deduced that \( \pi'[Q_p \cap Q_q] \leq \Lambda - 1 \), and

\[
\min\{\pi_p, \pi_q\} > \Lambda - 1 \geq \max\{\pi'[Q_p \triangle Q_q], \pi'[Q_p \cap Q_q]\} = \pi'[Q_p \cup Q_q] = \max\{\pi_p, \pi_q\},
\]

yielding a contradiction to Proposition 2 with \( S = \{p, q\} \). So \( \pi[Q_p \triangle Q_q] = \Lambda \), proving the proposition. \( \square \)

By Claim 4, suppose without loss generality that switching players 1 and \( \ell \) have their \( \pi \)-routes \( Q_1 \) and \( Q_\ell \) jointly cover \( R \). Intuitively, this coverage of \( R \) is likely to motivate a bilateral deviation by \( \{1, \ell\} \). To strengthen such an incentive, we choose such pair of players 1, \( \ell \) with \( Q_1 \) and \( Q_\ell \) being as long as possible. To be more specific,

\[
Q_1 \cup Q_\ell = R \iff E(Q_1 \cap Q_\ell) = \emptyset, \quad \text{and} \quad (4.3)
\]

\[
|E(Q_1)| + |E(Q_\ell)| = \min\{|E(Q_p)| + |E(Q_q)| \mid p, q \in [\ell], Q_p \cup Q_q = R\}. \quad (4.4)
\]

The minimality of \( |E(Q_1)| + |E(Q_\ell)| \) and the coverage \( Q_1 \cup Q_\ell = R \) guarantee that neither \( Q_1 \) nor \( Q_\ell \) is a proper subpath of any switching player’s \( \pi \)-route.

**Claim 6.** \( Q_1 \not\subset Q_j \) and \( Q_\ell \not\subset Q_j \) for all \( j = 2, 3, \ldots, \ell - 1 \). \( \square \)

Without loss of generality suppose \( \pi[Q_1] \leq \pi[Q_\ell] \). As \( Q_1 \cup Q_\ell = R \), to prevent joint deviation of players 1 and \( \ell \), their alternative paths bear very high \( \pi \)-loads as specified below.

**Claim 7.** \( \Lambda - 1 \leq \pi[Q_1] \leq \pi[Q_\ell] = \Lambda \).

**Proof.** Note from (4.3) that \( \pi[Q_1 \cap Q_\ell] = \pi[\emptyset] = 0 < 2\Lambda^* + 1 \leq \Lambda - 2 \), where the last inequality follows from \( \Lambda \geq 2\Lambda^* + 1 \) in Assumption 1. Then Proposition 5 enforces \( \pi[Q_\ell] \geq \pi[Q_1 \triangle Q_\ell] = \Lambda \).

If \( \pi[Q_1] < \Lambda - 1 \), then \( Q_1 \supseteq Q_\ell \) experiences \( \pi \)-load \( \pi[Q_1] \geq \pi[Q_\ell] \geq \Lambda > \pi[Q_1] + 1 \), showing a contradiction to Proposition 3. Hence \( \pi[Q_1] \geq \Lambda - 1 \).

Henceforth, let \( e \in Q_\ell \subseteq Q_1 \) denote a fixed most congested link with \( \pi[e] = \Lambda \geq 2\Lambda^* + 1 \), and let \( e' \in Q_1 \subseteq Q_\ell \) denote a fixed link with \( \pi[e'] \geq \Lambda - 1 \geq 2\Lambda^* \). See Figure 4(a).
4.2.3 Trilateral deviations

At this point, considering the most congested link \( e \) with \( \pi[e] = \Lambda \geq 2\Lambda^* + 1 \), the (second) most congested link \( e' \) with \( \pi[e'] \geq \Lambda - 1 \), and the pair of switching players \( 1, \ell \) is not enough for us to derive a contradiction. To identify more candidates for coalition \( S \), we need more links of high \( \pi \)-loads (see part I and the link set \( F \) over there), as well as switching players covering these links (see part II and the player set \( K \) over there). The final contradiction (trilateral deviation) is reached in part III.

I. Highly congested links  
Reasonably, we decrease the “threshold” \( \Lambda - 1 \) of \( \{e, e'\} \) a little bit to \( \Lambda - 2 \), and investigate its superset

\[
F \equiv \{ f \in E(\bar{Q}_1 \cup \bar{Q}_\ell) \mid \pi[f] \geq \Lambda - 2 \}. 
\]

**Claim 8.** \( \{e, e'\} \subseteq F \subseteq \bar{Q}_1 \cup \bar{Q}_\ell \), \( \min\{\pi[f] \mid f \in F\} \geq 2\Lambda^* - 1 \), and for any links \( f_1, f_2, f_3 \in F \), their total \( \pi \)-load is very high such that

\[
(i) \delta(f_1, f_2, f_3) \equiv (\pi[f_1] + \pi[f_2] + \pi[f_3]) - 3(2\Lambda^* - 1) \geq 0, \\
(ii) \delta(f_1, f_2, f_3) = 0 \text{ if and only if } \pi[f_1] = \pi[f_2] = \pi[f_3] = \Lambda - 2 = 2\Lambda^* - 1. 
\]

As depicted in Figure 4(a), let \( R_1 \) and \( R_2 \) denote the \( t_1-s_\ell \) path and \( t_\ell-s_1 \) path on \( R \) separated by the link-disjoint paths \( \bar{Q}_1 \) and \( \bar{Q}_\ell \) (the \( \pi^* \)-routes of switching players 1 and \( \ell \)) such that \( R \) is link-disjoint union of \( R_1, \bar{Q}_1, R_2, \bar{Q}_\ell \), and the four paths are on \( R \) in cyclic order. Let shortest subpaths \( H_1 \) and \( H_\ell \) of \( \bar{Q}_1 \) and \( \bar{Q}_\ell \) jointly covering all links in \( F \) (cf. Figure 4(b)) be specified as follows:

**Claim 9.** \( F \subseteq H_1 \cup H_\ell \subseteq \bar{Q}_1 \cup \bar{Q}_\ell \), where for \( j = 1, \ell \), \( H_j \) is the shortest subpath of \( \bar{Q}_j \) containing \( F \cap \bar{Q}_j \), the end links \( g_j \) and \( h_j \) of \( H_j \) belong to \( F \), and \( \pi(\bar{Q}_j \setminus E(H_j)) \leq \Lambda - 3. \)

In search for more candidate players for coalition \( S \), we wish to find switching players whose \( \pi \)-routes cover highly congested links as many as possible, because these players are more likely to be instable. If some player has its \( \pi \)-route contain \( H_1 \cup H_\ell \), and thus all links of \( F \), then it would be able to form a coalition with players 1 and \( \ell \), and deviate from \( \pi \) for lower load, contradicting the stability of \( \pi \). The next claim and its proof give a formal discussion.

**Claim 10.** There does not exist player \( t \in [m] \) such that \( H_1 \cup H_\ell \subseteq \bar{Q}_t \).
Proof. Suppose otherwise: \( H_1 \cup H_\ell \subseteq Q_t \) for some \( t \in [m] \), as depicted in Figure 4(b). It follows from Claim 9 that
\[
Q \equiv \bar{Q}_t \cap (Q_1 \cup Q_\ell) \subseteq (Q_1 \cup Q_\ell) \setminus E(H_1 \cup H_\ell) \text{ bears } \pi\text{-load } \pi[Q] \leq \Lambda - 3. \tag{4.5}
\]
So \( \pi[Q_1 \cap Q_\ell] \leq \pi[Q] \leq \Lambda - 3 \), and Proposition 5 enforces \( \pi[Q_1 \triangle Q_\ell] = \Lambda \). As \( E(\bar{Q}_1 \cap \bar{Q}_\ell) = \emptyset \), we see that \( E(Q_1 \triangle Q_\ell \cap Q_t) \subseteq Q_t \cap Q_t \subseteq Q \) bears \( \pi\)-load at most \( \Lambda - 3 \) by (4.5). Thus
\[
\Lambda = \pi[Q_1 \triangle Q_\ell] = \pi[Q_1 \triangle Q_\ell \cap Q_t] \leq \pi[Q] \leq \pi[\ell]. \tag{4.6}
\]
Observe that link \( e \) with \( \pi[e] = \Lambda \) belongs to \( F \cap \bar{Q}_\ell \cap Q_1 \subseteq H_\ell \subseteq Q_t \) (see Figure 4), giving
\[
\pi_j = \pi[Q_j] \geq \pi[e] = \Lambda \text{ for } j = 1, t. \tag{4.7}
\]

Comparing routing \( \pi' \equiv \pi^{(1, \ell, t)} \) and routing \( \pi \) (cf. Figure 4(b)), we see that
\[
\pi'[f] = \pi[f] - 1 \text{ for all } f \in \bar{Q}_1 \triangle Q_\ell \triangle Q_t, \text{ and } \pi'[f] = \pi[f] + 1 \text{ for all } f \in Q.
\]
Since \( E(\bar{Q}_1 \cap \bar{Q}_\ell) = \emptyset \), we deduce that \( E(\bar{Q}_1 \cup \bar{Q}_\ell \cup \bar{Q}_t) \) is the disjoint union of \( E(\bar{Q}_1 \triangle Q_\ell \triangle Q_t) \) and \( Q \). By (4.5) we obtain \( \pi_j' \leq \pi'\left( Q_1 \cup Q_\ell \cup Q_\ell\right) \leq \Lambda - 1 \) for \( j = 1, \ell, t \), which along with (4.6) and (4.7) implies a violation of Proposition 2 by \( S = \{1, \ell, t\} \).

Since \( \pi[e] + \pi[e'] \geq 4\Lambda^* + 1 \), as remarked at the end of Section 4.2.2, \( \{e, e'\} \subseteq Q_1 \) for some switching player \( t \). If \( |F| \leq 2 \), then \( F = \{e, e'\} \) by Claim 8, and the minimality of \( H_1, H_\ell \) in Claim 9 implies \( E(H_1 \cup H_\ell) = \{e, e'\} \subseteq Q_t \), implying a contradiction to Claim 10. Thus
\[
|F| \geq 3. \tag{4.8}
\]

For any \( f_1, f_2, f_3 \in F \), the big sum of \( \pi[f_1], \pi[f_2] \) and \( \pi[f_3] \) implied by \( \delta(f_1, f_2, f_3) \geq 0 \) in Claim 8 suggests us to look for a switching player whose \( \pi\)-route could cover \( f_1, f_2, f_3 \) whenever possible. Intuitively, there are \( \ell \) switching players in total. If every switching player has its \( \pi\)-route avoid, and thus its \( \pi^*\)-route contain, one of \( f_1, f_2, f_3 \), then \( \ell \leq \pi^*[f_1] + \pi^*[f_2] + \pi^*[f_3] \leq 3\Lambda^* \) would imply that \( \pi[f_1] + \pi[f_2] + \pi[f_3] \) is not large enough to guarantee \( \delta(f_1, f_2, f_3) > 0 \). The following proposition makes the idea precise.

**Proposition 11.** Given any triple of distinct links \( f_1, f_2, f_3 \in F \),

(i) if \( \pi\left[ Q_j \cap \{f_1, f_2, f_3\}\right] \leq 2 \) for every \( j \in [\ell - 1]\setminus\{1\} \), then \( \pi[f_1] = \pi[f_2] = \pi[f_3] = \Lambda - 2 \);

(ii) if \( \pi[f_1] \geq \Lambda - 1 \), then there exists some \( j \in [\ell - 1]\setminus\{1\} \) such that \( \{f_1, f_2, f_3\} \subseteq Q_j \).

**Proof.** It suffices to prove (i). An equivalent condition of (i) reads: for every switching player \( j \neq 1, \ell, \{Q_j \cap \{f_1, f_2, f_3\}\} \geq 1 \), implying
\[
\left| Q_j \cap \{f_1, f_2, f_3\} \right| - \left| \bar{Q}_j \cap \{f_1, f_2, f_3\} \right| \leq \left| \bar{Q}_j \cap \{f_1, f_2, f_3\} \right| \text{ for all } j \in [\ell - 1]\setminus\{1\}. \tag{4.9}
\]
Since \( \pi^*[f_i] \leq \Lambda^*, i = 1, 2, 3 \), from the difference between \( \pi \) and \( \pi^* \) in Assumption 1, we derive
\[
\begin{align*}
\delta(f_1, f_2, f_3) + 3\Lambda^* - 3 &= (\pi[f_1] + \pi[f_2] + \pi[f_3]) - 3\Lambda^* \\
&\leq (\pi[f_1] + \pi[f_2] + \pi[f_3]) - (\pi^*[f_1] + \pi^*[f_2] + \pi^*[f_3]) \\
&= \sum_{j=1}^{\ell} (\pi[f_j] - \pi^*[f_j]).
\end{align*}
\]
It is clear from \( E(\bar{Q}_1 \cap \bar{Q}_\ell) = \emptyset \) that each of \( f_1, f_2, f_3 \in F \subseteq \bar{Q}_1 \cup \bar{Q}_\ell \) is contained in exactly one of \( Q_1 \) and \( Q_\ell \), and in exactly one of \( Q_1 \) and \( Q_\ell \), implying
\[
\begin{align*}
\sum_{j=1,\ell} \left| Q_j \cap \{f_1, f_2, f_3\} \right| &= \sum_{i=1,2,3} (\left| Q_1 \cap \{f_i\} \right| + \left| Q_\ell \cap \{f_i\} \right|) = 3; \\
\sum_{j=1,\ell} \left| \bar{Q}_j \cap \{f_1, f_2, f_3\} \right| &= \sum_{i=1,2,3} (\left| \bar{Q}_1 \cap \{f_i\} \right| + \left| \bar{Q}_\ell \cap \{f_i\} \right|) = 3. \tag{4.10}
\end{align*}
\]
Hence $\sum_{j=1,\ell} |Q_j \cap \{f_1, f_2, f_3\}| - \sum_{j=1,\ell} |\overline{Q}_j \cap \{f_1, f_2, f_3\}| = 3 - 3 = 0$, and
\[ \delta(f_1, f_2, f_3) + 3\Lambda^* - 3 \leq \sum_{j=2}^{\ell-1} (|Q_j \cap \{f_1, f_2, f_3\}| - |\overline{Q}_j \cap \{f_1, f_2, f_3\}|) \]

It follows from (4.9) that
\[ \delta(f_1, f_2, f_3) + 3\Lambda^* - 3 \leq \sum_{j=1,\ell} |Q_j \cap \{f_1, f_2, f_3\}| \leq \pi'[f_1] + \pi'[f_2] + \pi'[f_3] - \sum_{j=1,\ell} |\overline{Q}_j \cap \{f_1, f_2, f_3\}| \]

Using (4.10), we obtain $\delta(f_1, f_2, f_3) + 3\Lambda^* - 3 \leq \pi'[f_1] + \pi'[f_2] + \pi'[f_3] - 3 \leq 3\Lambda^* - 3$, which together with $\delta(f_1, f_2, f_3) \geq 0$ in Claim 8(ii) implies $\delta(f_1, f_2, f_3) = 0$. Now statement (i) follows from Claim 8(ii). The proposition is proved. \hfill \Box 

The combination of Proposition 11 and Claim 10 is often used to derive more highly congested links in $F$; particularly, we show $|F \cap H_1| \geq 2$ as an improvement over $|F| \geq 3$ in (4.8).

**Claim 12.** The end links $g_1$ and $h_1$ of $H_1$ are different, and both belong to $F \cap \overline{Q}_1$ (cf. Figure 5(a,b)). Moreover $\pi[\overline{Q}_1 \setminus E(H_1)] \leq \Lambda - 3$.

**Proof.** Recall from Claim 8 that $e' \in \overline{Q}_1 \cap F$ bears load $\pi[e'] \geq \Lambda - 1$. If $\overline{Q}_1 \cap F$ consists of $e'$, then, $\{e'\} = E(H_1)$ by Claim 9, the end links $g_e$ of $h_e$ of path $H_e$ are different by $F \subseteq H_1 \cup H_\ell$ in Claim 9 and $|F| \geq 3$. Applying Proposition 11(ii) with $e', g_e, h_e$ in place of $f_1, f_2, f_3$, respectively, we obtain a player $t \in [\ell - 1] \setminus \{1\}$ whose $\pi$-route $Q_t$ contains $\{e', g_e, h_e\}$. Since $H_1 \cup H_\ell \not\subseteq Q_t$ by Claim 10, it must be the case that $Q_t$ is a subpath of $Q_\ell$ (cf. Figure 4), which contradicts Claim 6. Therefore $|F \cap \overline{Q}_1| \geq 2$. The statement follows from Claim 9. \hfill \Box 

**II. Heavily loaded players** Since Claim 10 has excluded the possibility for us to find any candidate player of $S$ whose $\pi$-route covers all links of $F$ in way of covering $H_1 \cup H_\ell$, the next best option is to look for some switching player whose $\pi$-route could cover $H_1$ and part of $H_\ell$. A starting point is considering the set $K$ of switching players whose $\pi$-routes go through the end links $g_1, h_1$ of $H_1$ and some link on $Q_\ell$ of load at least $\Lambda - 2$:
\[ K \equiv \{ j \in [\ell] \mid \{g_1, h_1\} \subseteq Q_j \text{ and } Q_j \cap \overline{Q}_\ell \cap F \neq \emptyset \}. \]

Obviously, $1, \ell \not\in K$ as $\{g_1, h_1\} \subseteq \overline{Q}_1$ by Claim 9 and $E(\overline{Q}_1 \cap \overline{Q}_\ell) = \emptyset$ by (4.3). Note from Claims 8 and 12 that $e, g_1, h_1 \in F$. Since $\pi[e] = \Lambda$, we may apply Proposition 11(ii) with $e, g_1, h_1$ in place of $f_1, f_2, f_3$, and obtain the existence of a player $j \in [\ell - 1] \setminus \{1\}$ whose $\pi$-route $Q_j$ contains $e, g_1, h_1$. It is easy to see that this player $j$ belongs to $K$ as $e \in Q_\ell$.

**Claim 13.** $\emptyset \neq K \subseteq [\ell - 1] \setminus \{1\}$. Moreover, for every player $j \in K$,

(i) its $\pi$-route $Q_j$ contains $H_1$ as a subpath;  
(ii) $\pi[Q_j \cap \overline{Q}_1] \leq \Lambda - 3$;  
(iii) $Q_j \cap \overline{Q}_\ell$ is a path.

**Proof.** If $H_1 \not\subseteq Q_j$ for some $j \in K$, then $\{g_1, h_1\} \subseteq Q_j$ enforces $Q_j$ to contain $\{g_1, h_1\} \cup R_2 \cup \overline{Q}_\ell \cup R_1$ and thus properly contain $Q_1$ (cf. Figure 4), contradicting $Q_1 \not\subseteq Q_j$ in Claim 6. So $H_1 \subseteq Q_j$.

For every $j \in K$, we see that $E(\overline{Q}_1 \cap \overline{Q}_j) = \overline{Q}_1 \setminus E(Q_j) \subseteq \overline{Q}_1 \setminus E(H_1)$ bears $\pi$-load at most $\Lambda - 3$ by Claim 12.

For any $j \in K$, the definition of $K$ says $E(Q_j \cap \overline{Q}_\ell) \neq \emptyset$. Since $Q_j$ cannot properly contain $Q_j$ by Claim 6, we see that $Q_j$ must intersect $\overline{Q}_\ell$ “continuously” at a single path in stead of two or more. \hfill \Box
Roughly, to choose a candidate player of \( S \) from \( K \), it is natural to think of a player in \( K \), say player 2, whose \( \pi \)-route \( Q_2 \) has a longest intersection with \( Q_\ell \), which may imply more links of \( F \) covered by \( Q_2 \) and hence high incentive for player 2 to deviate together with other players. Since \( H_1 \subseteq Q_2 \) by Claim 13(i), symmetry allows us to assume that path \( R_2 \) is internally contained by \( Q_2 \) (see Figures 4(a) and 5(a)).

![Figure 5: Trilateral deviations](image)

**Claim 14.** \( 2 \in K, |E(Q_2 \cap \bar{Q}_\ell)| = \max \{|E(Q_j \cap \bar{Q}_\ell)| \mid j \in K\}, H_1 \subseteq Q_2, \pi[\bar{Q}_1 \cap \bar{Q}_2] \leq \Lambda - 3, \) and \( \pi[\bar{Q}_1 \triangle Q_2] = \Lambda \).

**Proof.** \( \pi[\bar{Q}_1 \cap \bar{Q}_2] \leq \Lambda - 3 \) follows from Claim 13(ii), and subsequently \( \pi[\bar{Q}_1 \triangle Q_2] = \Lambda \) follows from Proposition 5.

Next we investigate the possibility of trilateral deviation by players 1, 2, \( \ell \). To prevent their coalitional deviating, \( Q_\ell \cap Q_2 \) has to contain some link of \( \pi \)-load at least \( \Lambda - 1 \).

**Claim 15.** \( \pi[Q_\ell \cap Q_2] \geq \Lambda - 1 \).

**Proof.** Comparing routing \( \pi \) (see Figure 5(a)) and routing \( \pi' = \pi^{1,2,\ell} \) (see Figure 5(b)), we observe that

\[
\begin{align*}
\pi'[f] &= \pi[f] - 1 \text{ for each link } f \in G_1 \equiv \bar{Q}_1 \triangle \bar{Q}_\ell \triangle \bar{Q}_2 \Rightarrow \pi'[G_1] \leq \Lambda - 1; \\
\pi'[f] &= \pi[f] + 1 \text{ for each link } f \in G_2 \equiv \bar{Q}_1 \cap \bar{Q}_2 \text{ and all } f \in G_3 \equiv \bar{Q}_\ell \cap \bar{Q}_2. 
\end{align*}
\]

(4.11)

Then from \( \pi[\bar{Q}_1 \cap \bar{Q}_2] \leq \Lambda - 3 \) in Claim 14, we derive \( \pi'[G_2] \leq \Lambda - 3 + 1 = \Lambda - 2 \).

If the claim does not hold, i.e., \( \pi[G_3] \leq \Lambda - 2 \), then from (4.11) it follows that \( \pi'[G_3] = \pi[G_3] + 1 \leq \Lambda - 1 \), and in turn from \( G_1 \cup G_2 \cup G_3 = \bar{Q}_1 \cup \bar{Q}_\ell \cup \bar{Q}_2 \) it follows that

\[
\pi'_j \leq \max \{\pi'[G_1], \pi'[G_2], \pi'[G_3]\} \leq \Lambda - 1 \text{ for } j = 1, 2, \ell.
\]

(4.12)

Moreover, from \( \pi[\bar{Q}_1 \triangle \bar{Q}_2] = \Lambda \) in Claim 14 and \( \pi[G_3] \leq \Lambda - 2 \), we deduce that \( \pi[\bar{Q}_1 \triangle \bar{Q}_2 \setminus E(G_3)] = \Lambda \). From \( Q_1 \subseteq \bar{Q}_\ell \) in (4.3), we get \( E(Q_1 \triangle Q_2) \setminus E(G_3) \subseteq E(Q_\ell) \), and in turn

\[
\pi_\ell = \pi[Q_\ell] \geq \pi[\bar{Q}_1 \triangle \bar{Q}_2 \setminus E(G_3)] = \Lambda.
\]

(4.13)

As \( e \in \bar{Q}_\ell \) with \( \pi[e] = \Lambda \) cannot be contained in \( G_3 \) with \( \pi[G_3] \leq \Lambda - 2 \), we have \( e \in E(Q_\ell) \setminus E(G_3) \subseteq Q_\ell \cap Q_2 \subseteq Q_1 \cap Q_2 \), saying \( \pi_j \geq \pi[e] = \Lambda \) for \( j = 1, 2 \), which in combination with (4.13) and (4.12) gives \( \pi_j = \Lambda > \Lambda - 1 \geq \pi'_j \) for \( j = 1, 2, \ell \), a contradiction to Proposition 2 with \( S = \{1, 2, \ell\} \).

\[\square\]
Recall from Assumption 1 that \( \Lambda \geq 2\Lambda^* + 1 \). Hence Claim 15 implies \( \pi(\bar{Q}_\ell \cap Q_2) \geq \Lambda - 1 \geq 2\Lambda^* \geq 2 \), in particular \( E(\bar{Q}_\ell \cap Q_2) \neq \emptyset \). It follows from Claim 13(iii) that \( Q_2 \) has a unique end, say \( t_2 \), in \( \bar{Q}_\ell \) such that \( Q_2 \cap Q_\ell \) is \( s.t_2 \) path avoiding node \( t_\ell \) (see Figure 5(a)).

Moreover, by Claim 15 we can take link \( g \in Q_\ell \cap Q_2 \) of load \( \pi(g) \geq \Lambda - 1 \) such that, on path \( Q_\ell \cap Q_2 \), link \( g \) and node \( t_2 \) (end node of \( Q_2 \) in \( \bar{Q}_\ell \)) are as close to each other as possible, which assures the validity of the following claim.

**Claim 16.** Subpath \( P \) of \( Q_\ell \cap Q_2 \) between link \( g \) and node \( t_2 \) bears load \( \pi[P] \leq \Lambda - 2 \). (See Figure 5(a,b).) \( \square \)

From Claim 12, we see that \( \{g, g_1, h_1\} \subseteq F \) and further that Proposition 11(ii) applies with \( g, g_1, h_1 \) in place of \( f_1, f_2, f_3 \), respectively. Hence there exists a switching player in \( [\ell - 1] \setminus \{1\} \) whose \( \pi \)-route goes through the three links \( g, g_1, h_1 \). We may assume that this switching player is 3, because \( g \in Q_2 \) means that \( Q_2 \), the \( \pi \)-route of player 2, does not contain \( g \). The definition of player set \( K \) and Claim 13 imply the following.

**Claim 17.** \( 3 \in K, g \in Q_3, H_1 \subseteq Q_3 \), and \( \pi(\bar{Q}_1 \cap \bar{Q}_3) \leq \Lambda - 3 \). \( \square \)

The fact that \( H_1 \subseteq Q_3, g \in Q_\ell \cap Q_2 \cap Q_3 \), and the maximality of \( |E(Q_2 \cap Q_\ell)| \) in Claim 14 enforce that \( Q_3 \) contains \( R_1 \) as an internal subpath (see Figures 4(a) and 5(a)). So \( Q_2 \cup Q_3 \cup P = R \), giving \( Q_1 \cup Q_2 \cup Q_3 = R \) as \( P \subseteq \bar{Q}_\ell \subseteq Q_1 \).

By Claims 14 and 17, \( H_1 \subseteq Q_2 \cap Q_3 \) implies that in \( \pi \) both players 2 and 3 go through all links on \( \bar{Q}_1 \) of load at least \( \Lambda - 2 \). Moreover, they jointly visit all links on \( \bar{Q}_\ell \) of load at least \( \Lambda - 1 \) (by Claim 16). In view that \( Q_\ell \subseteq Q_1 \) and \( \bar{Q}_1 \cap Q_j \) \( (j = 2, 3) \) bears \( \pi \)-load at most \( \Lambda - 3 \), the players 1, 2 and 3 would very likely form a coalition \( S \) to deviate if each of them experiences the highest \( \pi \)-load \( \Lambda \). Player 1 has already satisfied the condition because its \( \pi \)-route \( Q_1 \) contains the most congested link \( e \). The next two claims say that players 2 and 3 also satisfy the condition.

**Claim 18.** \( \pi_3 = \Lambda \).

**Proof.** Observe that \( Q_1 \triangle Q_2 \) contains both \( P \) and \( Q_1 \cap Q_3 \) (cf. Figure 5(c)), and \( E(Q_1 \triangle Q_2) \setminus E(P \setminus E(Q_1 \cap Q_3) \) is contained in \( Q_3 \) (cf. Figure 5(b)). Since \( \pi(Q_1 \triangle Q_2) = \Lambda \) by Claim 14 and \( \max\{\pi[P], \pi(Q_1 \cap Q_3)\} \leq \Lambda - 2 \) by Claims 16 and 17, we see that \( Q_3 \) bears a \( \pi \)-load at least \( \pi(Q_1 \triangle Q_2 \setminus E(P \setminus E(Q_1 \cap Q_3)) = \pi(Q_1 \triangle Q_2) = \Lambda \). \( \square \)

In the proof of the next claim, we apply Proposition 2 to \( S = \{1, 3\} \), where, as we have shown, players 1 and 3 experience highest \( \pi \)-load \( \Lambda \). The fact that they do not deviate jointly enforces that player 2 also suffers from highest load.

**Claim 19.** \( \pi_2 = \Lambda \).

**Proof.** For routing \( \pi' = \pi^{\{1, 3\}} \), by Proposition 2 with \( S = \{1, 3\} \) and \( \pi_1 = \Lambda = \pi_3 \), we obtain \( \pi'(Q_1 \cup Q_3) \geq \Lambda \). Observe that

\[
\pi'[f] = \pi[f] + 2 \text{ for all } f \in \bar{Q}_1 \cap Q_3, \text{ and } \pi'[f] = \pi[f] \text{ for all } f \in \bar{Q}_1 \triangle Q_3.
\] (4.14)

Since \( Q_1 \setminus E(H_1) \subseteq \bar{Q}_1 \setminus E(Q_3) \) by \( H_1 \subseteq Q_3 \) in Claim 17, it follows from Claim 12 that

\[
\pi'(Q_1 \setminus E(H_1)) \leq \pi[Q_1 \setminus E(H_1)] + 2 \leq \Lambda - 3 + 2 = \Lambda - 1.
\] (4.15)

As depicted in Figure 5(c), \( \bar{Q}_3 \cap P \subseteq Q_3 \setminus E(Q_1) \subseteq \bar{Q}_1 \triangle Q_3 \). (4.14) and Claim 16 imply that

\[
\pi'(Q_3 \cap P) = \pi[Q_3 \cap P] \leq \pi[P] \leq \Lambda - 2.
\] (4.16)
Notice that $\tilde{Q}_1 \cup \tilde{Q}_3$ is contained in the link-disjoint union of $Q_1 \setminus E(H_1)$, $Q_3 \cap P$, and $H_1 \cup (Q_1 \cap Q_2)$ (cf. Figure 5(c)). By $\pi'[\tilde{Q}_1 \cup \tilde{Q}_3] \geq \Lambda$, we deduce from (4.15) and (4.16) that $\pi'[H_1 \cup (Q_1 \cap Q_2)] = \pi'[\tilde{Q}_1 \cup \tilde{Q}_3] \geq \Lambda$.

As $H_1 \subseteq Q_3$, we see that $E(H_1) \cup E(Q_1 \cap Q_2)$ is disjoint from $Q_1 \cap Q_3$, and thus contained in $Q_1 \triangle Q_3$. It follows from (4.14) that $\pi[H_1 \cup (Q_1 \cap Q_2)] = \pi'[H_1 \cup (Q_1 \cap Q_2)] \geq \Lambda$.

Since $H_1 \subseteq Q_2$ by Claim 14, we have $H_1 \cup (Q_1 \cap Q_2) \subseteq Q_2$, which gives $\pi_2 = \pi[Q_2] \geq \pi[H_1 \cup (Q_1 \cap Q_2)] \geq \Lambda$ as desired.

\section{III. Trilateral deviation by bottleneck players} Having finished all necessary preparations, we are ready to show the final trilateral deviation by bottleneck players $1, 2, 3$ suffering from the maximum load $\pi_R = \Lambda$.

\textit{Proof of Theorem 4.2.} For the switching player $3 \in K$ (recall Claim 17), it can be deduced from Claim 13(iii) that $Q_3 \cap \bar{Q}_\ell$ is a path going through link $g$ and node $t_\ell$ (see Figure 5(b)). In routing $\pi' = \pi^{1.2.3}$ (see Figure 5(c)), players 2 and 3 have their $\pi'$-routes $Q_2$ and $Q_3$ intersect at $\bar{Q}_2 \cap Q_3 \subseteq P$ (possibly $\bar{Q}_2 \cap Q_3 = \emptyset$).

Since $E(Q_1 \cap Q_\ell) = \emptyset$, we see that $P \subseteq \bar{Q}_\ell$ is link-disjoint from $\pi'$-route $Q_1$ of player 1 and therefore is contained in its $\pi$-route $Q_1$. It follows that $\pi'[f] \leq \pi[f] + 1$ for all links $f \in P$ (cf. Figure 5(a,c)), which along with $\pi[P] \leq \Lambda - 2$ in Claim 16 gives

$$\pi'[\bar{Q}_2 \cap Q_3] \leq \pi'[P] \leq \pi[P] + 1 \leq \Lambda - 2 + 1 = \Lambda - 1. \quad (4.17)$$

It is obvious that $\pi'[f] = \pi[f] - 1$ for all links $f \in Q_1 \triangle \bar{Q}_2 \triangle Q_3$, implying

$$\pi'[Q_1 \triangle \bar{Q}_2 \triangle Q_3] \leq \Lambda - 1. \quad (4.18)$$

Observe that $\pi'[f] = \pi[f] + 1$ for all links $f \in Q_1 \cap (Q_2 \cup Q_3)$. Since by Claims 14 and 17 both $Q_1 \cap Q_2$ and $Q_1 \cap Q_3$ bear $\pi$-load at most $\Lambda - 3$, we deduce that

$$\pi'[Q_1 \cap (Q_2 \cup Q_3)] = \max\{\pi[Q_1 \cap Q_2], \pi[Q_1 \cap Q_3]\} + 1 \leq \Lambda - 2. \quad (4.19)$$

Since $Q_1 \cup Q_2 \cup Q_3$ is link-disjoint union of $Q_2 \cap Q_3$, $Q_1 \triangle Q_2 \triangle Q_3$ and $Q_1 \cap (Q_2 \cup Q_3)$, it follows from (4.17) – (4.19) that $\pi'_j \leq \Lambda - 1$ for $j = 1, 2, 3$.

Recall from $e \in Q_1$ and Claims 18 and 19 that $\pi_j = \Lambda$ for $j = 1, 2, 3$, which together with the upper bound $\Lambda - 1$ on $\pi'_1, \pi'_2, \pi'_3$ yields a violation of Proposition 2 by $S = \{1, 2, 3\}$. This final contradiction shows that Assumption 1 is incorrect, and therefore establishes Theorem 4.2.

\section{Conclusion}

In this paper we have studied the selfish ring routing game for load balancing that allows coalitions among self-interested players (SRLC). Our main results show that the $k$-SPoA of SRLC is bounded above by 2 for all $k \geq 3$, in contrast to its unbounded PoA = 2-SPoA = $n - 1$. This significant improvement on global efficiency is highly realizable in decentralized environments since players themselves are able to easily determine (say by enumeration) coalitions of size at most three whose deviation can make every member better off. This approach is particularly useful for large scale competitive games, where only small-scale communication and computation are realizable.

For future work, it is interesting to explore the weighted version of SRLC, where atomic selfish ring routing has to carry nonuniform traffic between different source-destination pairs. We believe that $k$-SPoA of SRLC with nonuniform traffic could also be upper bounded by some constant for all $k \geq 3$. Our preliminary study shows that 3-SPoA is at most 6 when
there are only two different weights. Other challenging direction is to investigate if the method could be extended to general networks, making good global balance via small coalitions and some other techniques.

References


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