Total Dual Integrality in Some Facility Location Problems

Xujin Chen

Institute of Applied Math., Chinese Academy of Sciences

Joint work with
Zhibin Chen, Wenan Zang

2013 Symposium on Graph Theory & its Applications
March 30 – 31, 2013, Tianjin, China
Outline

Facility Location
Uncapcitated facility location (UFL)
Prize-collecting facility location (PFL)

Dual integrality
Characterization and algorithm for PFL
Characterization and algorithm for UFL

Proofs
Total dual integrality of PFL
Total dual integrality of UFL
Given a set $F$ of facilities and a set $C$ of customers such that each facility $i$ has an opening cost $c_i$ and serving customer $j$ by facility $i$ incurs a cost $c_{ij}$, the Uncapacitated Facility Location (UFL) problem is to open a subset of facilities in $F$ and serve each customer by an open facility at minimum total cost.
**Uncapacitated Facility Location (UFL)**

**Given** a set $F$ of facilities and a set $C$ of customers such that each facility $i$ has an opening cost $c_i$ and serving customer $j$ by facility $i$ incurs a cost $c_{ij}$, the **Uncapacitated Facility Location (UFL)** problem is to open a subset of facilities in $F$ and serve each customer by an open facility at minimum total cost.

\[ G = (F \cup C, A) \]
Prize-Collecting Facility location (PFL)

**PFL**: “prize-collecting" version of the UFL [Baïou, Barahona’09]

**Given**: digraph $G = (V, A)$ with integer weight $w_x$ on each $x \in V \cup A$.
For $v \in V$, $w_v = \text{the profit made by opening a facility at } v$;
For $uv \in A$, $w_{uv} = \text{the profit made by serving customer } v \text{ with facility } u$

**Task**: select a subset of vertices, and then assign some (but not necessarily all) nonselected vertices to centers.

**Goal**: maximize the total opening and service profit.
Prize-Collecting Facility location (PFL)

**PFL:** “prize-collecting" version of the UFL [Baïou, Barahona’09]

**Given:** digraph \( G = (V, A) \) with integer weight \( w_x \) on each \( x \in V \cup A \).

For \( v \in V \), \( w_v \) = the profit made by opening a facility at \( v \);

For \( uv \in A \), \( w_{uv} \) = the profit made by serving customer \( v \) with facility \( u \)

**Task:** select a subset of vertices, and then assign some (but not necessarily all) nonselected vertices to centers.

**Goal:** maximize the total opening and service profit.

\[
\text{(PP) Maximize } \sum_{uv \in A} w_{uv}x_{uv} + \sum_{v \in V} w_v y_v \\
\text{subject to } \sum_{uv \in A} x_{uv} + y_v \leq 1 \quad \forall v \in V, \quad (1a) \\
\quad x_{uv} - y_u \leq 0 \quad \forall uv \in A, \quad (1b) \\
\quad x_{uv} \geq 0 \quad \forall uv \in A, \quad (1c) \\
\quad y_v \geq 0 \quad \forall v \in V. \quad (1d)
\]
Outline

**Facility Location**
Uncapcitated facility location (UFL)
Prize-collecting facility location (PFL)

**Dual integrality**
Characterization and algorithm for PFL
Characterization and algorithm for UFL

**Proofs**
Total dual integrality of PFL
Total dual integrality of UFL
A vertex of $G$ is a source (resp. sink) if $G$ has no arc entering (resp. leaving) it, and is mixed if it is neither a source nor a sink.

$$\hat{G} = \{\text{sources in } G\}$$
$$\tilde{G} = \{\text{mixed vertices in } G\}.$$
A vertex of $G$ is a *source* (resp. *sink*) if $G$ has no arc entering (resp. leaving) it, and is *mixed* if it is neither a source nor a sink.

Let $C$ be a cycle (not necessarily directed) in $G$.

- $C$ is *odd* if $|\tilde{C}| + |\hat{C}|$ is odd and *even* otherwise.
Definitions

A vertex of $G$ is a source (resp. sink) if $G$ has no arc entering (resp. leaving) it, and is mixed if it is neither a source nor a sink.

$$\dot{G} = \{\text{sources in } G\}$$
$$\tilde{G} = \{\text{mixed vertices in } G\}.$$

Let $C$ be a cycle (not necessarily directed) in $G$.

- $C$ is odd if $|\tilde{C}| + |\dot{C}|$ is odd and even otherwise.
- $G$ even if each cycle of $G$ is even.
Definitions

A vertex of $G$ is a source (resp. sink) if $G$ has no arc entering (resp. leaving) it, and is mixed if it is neither a source nor a sink.

$\hat{G} = \{\text{sources in } G\}$

$\tilde{G} = \{\text{mixed vertices in } G\}.$

Let $C$ be a cycle (not necessarily directed) in $G$.

$\triangleright$ $C$ is odd if $|\tilde{C}| + |\hat{C}|$ is odd and even otherwise.

$\triangleright$ $G$ even if each cycle of $G$ is even.

$\triangleright$ polyhedron $P$ is integral if each face of $P$ contains integral vectors, and is called a polytope if $P$ is bounded.

$\triangleright$ a polytope is integral if and only if its vertices are all integral.
**Prize-Collecting Facility location (PFL)**

**Given:** digraph $G = (V, A)$ with integer weight $w_x$ on each $x \in V \cup A$.

**Task:** select a subset of vertices, and assign some (but not necessarily all) nonselected vertices to centers.

**Goal:** maximize the total opening and service profit.

\[
(PP) \quad \text{Maximize} \quad \sum_{uv \in A} w_{uv} x_{uv} + \sum_{v \in V} w_v y_v \\
\text{subject to} \quad \sum_{uv \in A} x_{uv} + y_v \leq 1 \quad \forall \ v \in V, \quad (1a) \\
x_{uv} - y_u \leq 0 \quad \forall \ uv \in A, \quad (1b) \\
x_{uv} \geq 0 \quad \forall \ uv \in A, \quad (1c) \\
y_v \geq 0 \quad \forall \ v \in V. \quad (1d)
\]
Prize-Collecting Facility location (PFL)

**Given:** digraph $G = (V, A)$ with integer weight $w_x$ on each $x \in V \cup A$.

**Task:** select a subset of vertices, and assign some (but not necessarily all) nonselected vertices to centers.

**Goal:** maximize the total opening and service profit.

\[
\begin{align*}
\text{(PP)} & \quad \text{Maximize} \quad & \sum_{uv \in A} w_{uv}x_{uv} + \sum_{v \in V} w_vy_v \\
\text{subject to} & \quad & \sum_{uv \in A} x_{uv} + y_v \leq 1 \quad \forall \ v \in V, \quad (1a) \\
& \quad & x_{uv} - y_u \leq 0 \quad \forall \ uv \in A, \quad (1b) \\
& \quad & x_{uv} \geq 0 \quad \forall \ uv \in A, \quad (1c) \\
& \quad & y_v \geq 0 \quad \forall \ v \in V. \quad (1d)
\end{align*}
\]

Let $\pi(G)$ denote the linear system $(1a-d)$

Let $P(G) = \{(x, y) : (x, y) \text{ satisfies (1a-d)}\}$ denote the polytope defined by $\pi(G)$. 

Xujin Chen, Zhibin Chen, Wenan Zang
Let $C$ be a cycle (not necessarily directed) in $G$.

- $C$ is *odd* if $|\tilde{C}| + |\dot{C}|$ is odd and *even* otherwise.
- $G$ is *even* if each cycle of $G$ is even.
- A polytope is integral if and only if its vertices are all integral.

Let $\pi(G)$ denote the linear system (1a-d)

Let $P(G)$ denote the polytope defined by $\pi(G)$.

**Theorem** (Baïou & Barahona, 2009)

Let $G$ be the input digraph of the PFL. Then $P(G)$ is integral if and only if $G$ is even.
Algorithms for PFL

**Theorem (Baïou & Barahona, 2009)**

*Let $G$ be the input digraph of the PFL. Then $P(G)$ is integral if and only if $G$ is even.*
Algorithms for PFL

Theorem (Baïou & Barahona, 2009)
Let $G$ be the input digraph of the PFL. Then $P(G)$ is integral if and only if $G$ is even.

Theorem (Tardos 1986)
There exists a strongly polynomial-time algorithm for LP problems with $(0, \pm 1)$ constraint matrices.
Theorem (Baïou & Barahona, 2009)

Let $G$ be the input digraph of the PFL. Then $P(G)$ is integral if and only if $G$ is even.

Theorem (Tardos 1986)

There exists a strongly polynomial-time algorithm for LP problems with $(0, \pm 1)$ constraint matrices.

- an opt solution to (PP) can be found in strongly polynomial time;
- it can be further transformed into an optimal basic solution $(x^*, y^*)$ in strongly polynomial time [Luenberger & Ye, 2008];
Theorem (Baïou & Barahona, 2009)

Let $G$ be the input digraph of the PFL. Then $P(G)$ is integral if and only if $G$ is even.

Theorem (Tardos 1986)

There exists a strongly polynomial-time algorithm for LP problems with $(0, \pm 1)$ constraint matrices.

- an opt solution to (PP) can be found in strongly polynomial time;
- it can be further transformed into an optimal basic solution $(x^*, y^*)$ in strongly polynomial time [Luenberger & Ye, 2008];
- $P(G)$ is integral $\Rightarrow (x^*, y^*)$ is integral.
Algorithms for PFL

**Theorem (Baïou & Barahona, 2009)**

Let $G$ be the input digraph of the PFL. Then $P(G)$ is integral if and only if $G$ is even.

**Theorem (Tardos 1986)**

There exists a strongly polynomial-time algorithm for LP problems with $(0, \pm 1)$ constraint matrices.

- an opt solution to (PP) can be found in strongly polynomial time;
- it can be further transformed into an optimal basic solution $(x^\ast, y^\ast)$ in strongly polynomial time [Luenberger & Ye, 2008];
- $P(G)$ is integral $\Rightarrow (x^\ast, y^\ast)$ is integral.

**Cor [Baïou & Barahona, 2009]** $\exists$ a strongly polynomial-time algorithm for PFL if $G$ is even.
Algorithms for PFL

**Theorem** (Baïou & Barahona, 2009)

Let $G$ be the input digraph of the PFL. Then $P(G)$ is integral if and only if $G$ is even.

**Theorem** (Tardos 1986)

There exists a strongly polynomial-time algorithm for LP problems with $(0, \pm 1)$ constraint matrices.

- an opt solution to (PP) can be found in strongly polynomial time;
- it can be further transformed into an optimal basic solution $(x^*, y^*)$ in strongly polynomial time [Luenberger & Ye, 2008];
- $P(G)$ is integral $\Rightarrow (x^*, y^*)$ is integral.

**Cor** [Baïou & Barahona, 2009] $\exists$ a strongly polynomial-time algorithm for PFL if $G$ is even.

? Is there a combinatorial polynomial-time algorithm for PFL?
Linear system $Ax \leq b$ is *totally dual integral* (TDI) if the minimum in
\[
\max \{ w^T x : Ax \leq b \} = \min \{ y^T b : y^T A = w^T, y \geq 0 \}
\]
has an integral optimal solution, for every integral vector $w$ for which
the optimum is finite.
Total Dual Integrality in PFL

Linear system $Ax \leq b$ is totally dual integral (TDI) if the minimum in
$$\max\{w^T x : Ax \leq b\} = \min\{y^T b : y^T A = w^T, \ y \geq 0\}$$
has an integral optimal solution, for every integral vector $w$ for which
the optimum is finite.

**Th. [Edmonds & Giles ’77]** If $Ax \leq b$ is TDI and $b$ is integral, then
{$x : Ax \leq b$} is integral.

**Theorm (C, Chen, Zang, 2012)**

Let $G = (V, A)$ be a digraph. Then the following are equivalent:

(i) $G$ is even;
(ii) $P(G)$ is integral; and
(iii) $\pi(G)$ is TDI.

Moreover, if $G$ is even, then an optimal solution to PFL can be found
in $O(|A|^2 \log^2 |A|)$ time.
Linear system $Ax \leq b$ is *totally dual integral* (TDI) if the minimum in

$$\max \{w^T x : Ax \leq b\} = \min \{y^T b : y^T A = w^T, y \geq 0\}$$

has an integral optimal solution, for every integral vector $w$ for which the optimum is finite.

**Th.** [Edmonds & Giles ’77] If $Ax \leq b$ is TDI and $b$ is integral, then \{\(x : Ax \leq b\)\} is integral.

**Theorm (C, Chen, Zang, 2012)**

Let $G = (V, A)$ be a digraph. Then the following are equivalent:

(i) $G$ is even;
(ii) $P(G)$ is integral; and
(iii) $\pi(G)$ is TDI.

Moreover, if $G$ is even, then an optimal solution to PFL can be found in $O(|A|^2 \log^2 |A|)$ time. ← **combinatorial algorithm!**
Suppose $\pi(G)$ is TDI, and $w \in \mathbb{Z}^{V \cup A}$. Then $\exists$ an integral optimal solution $(x^*, y^*)$ to (PP) and an integral optimal solution $(\alpha^*, \beta^*)$ to

$$(PD) \quad \text{Minimize } \sum_{v \in V} \alpha_v$$

subject to

$\alpha_v + \beta_{uv} \geq w_{uv} \quad \forall \ uv \in A,$  \hspace{1cm} (2a)

$\alpha_u - \sum_{uv \in A} \beta_{uv} \geq w_u \quad \forall \ u \in V,$  \hspace{1cm} (2b)

$\beta_{uv} \geq 0 \quad \forall \ uv \in A,$  \hspace{1cm} (2c)

$\alpha_v \geq 0 \quad \forall \ v \in V.$  \hspace{1cm} (2d)
Suppose $\pi(G)$ is TDI, and $w \in \mathbb{Z}^{V \cup A}$. Then exists an integral optimal solution $(x^*, y^*)$ to (PP) and an integral optimal solution $(\alpha^*, \beta^*)$ to (PD)

\[
\begin{align*}
\text{(PD)} & \quad \text{Minimize} & \sum_{v \in V} \alpha_v \\
& \quad \text{subject to} & \alpha_v + \beta_{uv} \geq w_{uv} & \forall uv \in A, \quad (2a) \\
& & \alpha_u - \sum_{uv \in A} \beta_{uv} \geq w_u & \forall u \in V, \quad (2b) \\
& & \beta_{uv} \geq 0 & \forall uv \in A, \quad (2c) \\
& & \alpha_v \geq 0 & \forall v \in V. \quad (2d)
\end{align*}
\]

- $x^*$ and $y^*$ are 0-1 vectors ......
Primal vs Dual

Suppose $\pi(G)$ is TDI, and $w \in \mathbb{Z}^{V \cup A}$. Then $\exists$ an integral optimal solution $(x^*, y^*)$ to (PP) and an integral optimal solution $(\alpha^*, \beta^*)$ to

(PD) Minimize $\sum_{v \in V} \alpha_v$

subject to $\alpha_v + \beta_{uv} \geq w_{uv}$ $\forall uv \in A$, $(2a)$

$\alpha_u - \sum_{uv \in A} \beta_{uv} \geq w_u$ $\forall u \in V$, $(2b)$

$\beta_{uv} \geq 0$ $\forall uv \in A$, $(2c)$

$\alpha_v \geq 0$ $\forall v \in V$. $(2d)$

$\blacktriangleright \alpha_v^* > 0 \Rightarrow \sum_{uv \in A} x_{uv}^* + y_v^* = 1$: open $v$ or serve $v$ by some other vertex $u$
Suppose $\pi(G)$ is TDI, and $w \in \mathbb{Z}^{V \cup A}$. Then \exists an integral optimal solution $(x^*, y^*)$ to (PP) and an integral optimal solution $(\alpha^*, \beta^*)$ to

(PD) Minimize $\sum_{v \in V} \alpha_v$

subject to

$\alpha_v + \beta_{uv} \geq w_{uv}$ \hspace{1cm} \forall uv \in A, \hspace{1cm} (2a)$

$\alpha_u - \sum_{uv \in A} \beta_{uv} \geq w_u$ \hspace{1cm} \forall u \in V, \hspace{1cm} (2b)$

$\beta_{uv} \geq 0$ \hspace{1cm} \forall uv \in A, \hspace{1cm} (2c)$

$\alpha_v \geq 0$ \hspace{1cm} \forall v \in V. \hspace{1cm} (2d)$

- $\alpha^*_v > 0 \Rightarrow \sum_{uv \in A} x^*_{uv} + y^*_v = 1$: open $v$ or serve $v$ by some other vertex $u$
- $\alpha^*_v = \text{the opening payment or the service payment at } v$
Suppose $\pi(G)$ is TDI, and $w \in \mathbb{Z}^{V \cup A}$. Then $\exists$ an integral optimal solution $(x^*, y^*)$ to (PP) and an integral optimal solution $(\alpha^*, \beta^*)$ to

\begin{align*}
\text{(PD) Minimize} & \quad \sum_{v \in V} \alpha_v \\
\text{subject to} & \quad \alpha_v + \beta_{uv} \geq w_{uv} \quad \forall uv \in A, \quad (2a) \\
& \quad \alpha_u - \sum_{uv \in A} \beta_{uv} \geq w_u \quad \forall u \in V, \quad (2b) \\
& \quad \beta_{uv} \geq 0 \quad \forall uv \in A, \quad (2c) \\
& \quad \alpha_v \geq 0 \quad \forall v \in V. \quad (2d)
\end{align*}

- $\alpha^*_v > 0 \Rightarrow \sum_{uv \in A} x^*_{uv} + y^*_v = 1$: open $v$ or serve $v$ by some other vertex $u$
- $\alpha^*_v$ = the opening payment or the service payment at $v$
- $y^*_u = 1 \Rightarrow \alpha^*_u - \sum_{uv \in A} \beta^*_{uv} = w_u$
Suppose $\pi(G)$ is TDI, and $w \in \mathbb{Z}^{V \cup A}$. Then $\exists$ an integral optimal solution $(x^*, y^*)$ to (PP) and an integral optimal solution $(\alpha^*, \beta^*)$ to (PD)

Minimize $\sum_{v \in V} \alpha_v$

subject to $\alpha_v + \beta_{uv} \geq w_{uv}$ $\forall uv \in A$, (2a)

$\alpha_u - \sum_{uv \in A} \beta_{uv} \geq w_u$ $\forall u \in V$, (2b)

$\beta_{uv} \geq 0$ $\forall uv \in A$, (2c)

$\alpha_v \geq 0$ $\forall v \in V$. (2d)

- $\alpha^*_v > 0 \Rightarrow \sum_{uv \in A} x^*_{uv} + y^*_v = 1$: open $v$ or serve $v$ by some other vertex $u$
- $\alpha^*_v = \text{the opening payment or the service payment at } v$
- $y^*_u = 1 \Rightarrow \alpha^*_u - \sum_{uv \in A} \beta^*_{uv} = w_u \Rightarrow \alpha^*_u - \sum_{v: x^*_v=1} \beta^*_{uv} = w_u$
Suppose $\pi(G)$ is TDI, and $w \in \mathbb{Z}^{V \cup A}$. Then $\exists$ an integral optimal solution $(x^*, y^*)$ to (PP) and an integral optimal solution $(\alpha^*, \beta^*)$ to

(PD) Minimize $\sum_{v \in V} \alpha_v$

subject to

$\alpha_v + \beta_{uv} \geq w_{uv} \quad \forall \ uv \in A,$  \hspace{0.5cm} (2a)

$\alpha_u - \sum_{uv \in A} \beta_{uv} \geq w_u \quad \forall \ u \in V,$  \hspace{0.5cm} (2b)

$\beta_{uv} \geq 0 \quad \forall \ uv \in A,$ \hspace{0.5cm} (2c)

$\alpha_v \geq 0 \quad \forall \ v \in V.$ \hspace{0.5cm} (2d)

$\triangleright \ \alpha_v^* > 0 \Rightarrow \sum_{uv \in A} x_{uv}^* + y_v^* = 1$: open $v$ or serve $v$ by some other vertex $u$

$\triangleright \ \alpha_v^* = \text{the opening payment or the service payment at } v$

$\triangleright \ y_u^* = 1 \Rightarrow \alpha_u^* - \sum_{uv \in A} \beta_{uv}^* = w_u \Rightarrow \alpha_u^* - \sum_{v:x_{uv}^*=1} \beta_{uv}^* = w_u$

as $\varphi(v) \neq u \iff x_{uv}^* = 0 \Rightarrow x_{uv}^* - y_u^* < 0 \Rightarrow \beta_{uv}^* = 0$
Suppose $\pi(G)$ is TDI, and $w \in \mathbb{Z}^{V \cup A}$. Then there exists an integral optimal solution $(x^*, y^*)$ to (PP) and an integral optimal solution $(\alpha^*, \beta^*)$ to

$$(PD) \quad \text{Minimize} \quad \sum_{v \in V} \alpha_v$$
$$\text{subject to} \quad \alpha_v + \beta_{uv} \geq w_{uv} \quad \forall uv \in A, \quad (2a)$$
$$\alpha_u - \sum_{uv \in A} \beta_{uv} \geq w_u \quad \forall u \in V, \quad (2b)$$
$$\beta_{uv} \geq 0 \quad \forall uv \in A, \quad (2c)$$
$$\alpha_v \geq 0 \quad \forall v \in V. \quad (2d)$$

- $\alpha_v^* > 0 \Rightarrow \sum_{uv \in A} x_{uv}^* + y_v^* = 1$: open $v$ or serve $v$ by some other vertex $u$
- $\alpha_v^* = \text{the opening payment or the service payment at } v$
- $y_u^* = 1 \Rightarrow \alpha_u^* - \sum_{uv \in A} \beta_{uv}^* = w_u \Rightarrow \alpha_u^* - \sum_v: x_{uv}^* = 1 \beta_{uv}^* = w_u$
- $x_{uv}^* = 1 \Rightarrow \alpha_v^* + \beta_{uv}^* = w_{uv}$
Min-max Relation

Theorem (C, Chen, Zang, 2012)

Let $G = (V, A)$ be a digraph. Then the following are equivalent:

(i) $G$ is even;
(ii) $P(G)$ is integral; and
(iii) $\pi(G)$ is TDI.

Moreover, if $G$ is even, an optimal solution to PFL can be found in $O(|A|^2 \log^2 |A|)$ time.

Corollary (C, Chen, Zang, 2012)

Let $G = (V, A)$ be a digraph. Then the minimum (integral) payment paid to the locations is equal to the maximum total profit made in PFL, for all integral $w \in \mathbb{Z}^{V \cup A}$, if and only if $G$ is even.
Outline

Facility Location
  Uncapcitated facility location (UFL)
  Prize-collecting facility location (PFL)

Dual integrality
  Characterization and algorithm for PFL
  Characterization and algorithm for UFL

Proofs
  Total dual integrality of PFL
  Total dual integrality of UFL
Uncapacitated Facility Location (UFL)

**Given** bipartite digraph $G = (F \cup C, A)$ and an integral nonnegative cost function $c$ defined on $F \cup A$, where $(F, C)$ is the bipartition of $G$ and all arcs of $G$ are directed from $F$ to $C$.

**UFL** is to open some facilities in $F$ to serve all customers in $C$ at minimum total cost, where opening a facility at $u \in F$ incurs a cost $c_u$, and using $u \in F$ to serve its neighbor $v \in C$ incurs a service cost $c_{uv}$.

**(UP)** Minimize $\sum_{uv \in A} c_{uv} x_{uv} + \sum_{u \in F} c_u y_u$

subject to

$\sum_{uv \in A} x_{uv} = 1$ \hspace{1cm} $\forall v \in C$, \hspace{1cm} (3a)

$y_u - x_{uv} \geq 0$ \hspace{1cm} $\forall uv \in A$, \hspace{1cm} (3b)

$x_{uv} \geq 0$ \hspace{1cm} $\forall uv \in A$, \hspace{1cm} (3c)

$y_u \geq 0$ \hspace{1cm} $\forall u \in F$. \hspace{1cm} (3d)
FL Polytope

\[(UP) \quad \text{Minimize} \quad \sum_{uv \in A} c_{uv}x_{uv} + \sum_{u \in F} c_u y_u \]

subject to

\[\sum_{uv \in A} x_{uv} = 1 \quad \forall v \in C, \quad (3a)\]

\[y_u - x_{uv} \geq 0 \quad \forall uv \in A, \quad (3b)\]

\[x_{uv} \geq 0 \quad \forall uv \in A, \quad (3c)\]

\[y_u \geq 0 \quad \forall u \in F. \quad (3d)\]

Let \(\sigma(G)\) denote the linear system (3a-d)
Let \(Q(G) = \{(x, y) : (x, y) \text{ satisfies (3a-d)}\}\) denote the polyhedron defined by \(\sigma(G)\), i.e., fractional UFL polyhedron of \(G\).

Let \(C' = \{v : v \in C & \deg_{G}(v) = 1\}\)
Let \(F' = \{u : u \in F & \exists v \in C' \text{ s.t. } uv \in A\}\)
Let \(G' = G \setminus (C' \cup F')\)
Integral FL Polytope

Let $\sigma(G)$ denote the linear system (3a-d)
Let $Q(G) = \{(x,y) : (x,y) \text{ satisfies (3a-d)}\}$ denote the polyhedron defined by $\sigma(G)$, i.e., fractional UFL polyhedron of $G$.
Let $C' = \{v : v \in C \& \deg_G(v) = 1\}$
Let $F' = \{u : u \in F \& \exists v \in C' \text{ s.t. } uv \in A\}$
Let $G' = G \setminus (C' \cup F')$

Theorem (Baïou & Barahona, 2009)

Let $G$ be the input digraph of UFL. Then $Q(G)$ is integral if and only if $G'$ is even.
Let $\sigma(G)$ denote the linear system (3a-d)
Let $Q(G) = \{(x,y) : (x,y) \text{ satisfies (3a-d)}\}$ denote the polyhedron defined by $\sigma(G)$, i.e., fractional UFL polyhedron of $G$.
Let $C' = \{v : v \in C \& \deg_G(v) = 1\}$
Let $F' = \{u : u \in F \& \exists v \in C' \text{ s.t. } uv \in A\}$
Let $G' = G \setminus (C' \cup F')$

**Theorem (Baïou & Barahona, 2009)**

Let $G$ be the input digraph of UFL. Then $Q(G)$ is integral if and only if $G'$ is even.

- $G'$ is even $\iff$ the length of each cycle in $G'$ is a multiple of 4.
- $\rightarrow$ a strongly poly-time algorithm for UFL when $G'$ is even

?? Is there a combinatorial poly-time algorithm for the problem?
Theorem (Baïou & Barahona, 2009)

Let $G$ be the input digraph of UFL. Then $Q(G)$ is integral if and only if $G'$ is even.

Theorem (C, Chen, Zang, 2012)

Let $G = (F \cup C, A)$ be a digraph. Then the following are equivalent:

(i) $G'$ is even;
(ii) $Q(G)$ is integral; and
(iii) $\sigma(G)$ is TDI.

Moreover, if $G'$ is even, then an optimal solution to UFL can be found in $O(|A|^2 \log^2 |A|)$ time (by a combinatorial algorithm).
Suppose $\sigma(G)$ is TDI. Then (UP) has an integral optimal solution $(x^*, y^*)$ and (UD) has an integral optimal solution $(\alpha^*, \beta^*)$.

(UD) \[
\text{Maximize} \quad \sum_{v \in C} \alpha_v \\
\text{subject to} \quad \alpha_v - \beta_{uv} \leq c_{uv} \quad \forall \ uv \in A, \quad (4a) \\
\sum_{uv \in A} \beta_{uv} \leq c_u \quad \forall \ u \in F, \quad (4b) \\
\beta_{uv} \geq 0 \quad \forall \ uv \in A. \quad (4c)
\]
Suppose $\sigma(G)$ is TDI. Then (UP) has an integral optimal solution $(x^*, y^*)$ and (UD) has an integral optimal solution $(\alpha^*, \beta^*)$.

(UD) Maximize $\sum_{v \in C} \alpha_v$

subject to $\alpha_v - \beta_{uv} \leq c_{uv}$ $\forall uv \in A$, (4a)

$\sum_{uv \in A} \beta_{uv} \leq c_u$ $\forall u \in F$, (4b)

$\beta_{uv} \geq 0$ $\forall uv \in A$. (4c)
Suppose $\sigma(G)$ is TDI. Then (UP) has an integral optimal solution $(x^*, y^*)$ and (UD) has an integral optimal solution $(\alpha^*, \beta^*)$.

(UD) Maximize $\sum_{v \in C} \alpha_v$

subject to $\alpha_v - \beta_{uv} \leq c_{uv}$ $\forall uv \in A$, (4a)

$\sum_{uv \in A} \beta_{uv} \leq c_u$ $\forall u \in F$, (4b)

$\beta_{uv} \geq 0$ $\forall uv \in A$. (4c)

$\star \alpha_v =$ price paid by customer $v \in C$

- $(x^*, y^*)$ is a 0-1 vector...
- $y_u^* = 1 \Rightarrow \sum_{uv \in A} \beta_{uv}^* = c_u \Rightarrow \sum_{v: x_{uv}^* = 1} \beta_{uv}^* = c_u$
- $x_{uv}^* = 1 \Rightarrow \alpha_v^* - \beta_{uv}^* = c_{uv}$
Min-max Relation

Theorem (C, Chen, Zang, 2012)

Let \( G = (F \cup C, A) \) be a digraph. Then the following are equivalent:

(i) \( G^* \) is even;
(ii) \( Q(G) \) is integral; and
(iii) \( \sigma(G) \) is TDI.

Moreover, if \( G' \) is even, then an optimal solution to UFL can be found in \( O(|A|^2 \log^2 |A|) \) time.

Corollary (C, Chen, Zang, 2012)

Let \( G = (F \cup C, A) \) be the input digraph of UFL. Then the minimum total opening and connection cost is equal to the maximum (integral) total price the customers are willing to pay, for all integral \( c \in \mathbb{Z}_{FUA}^+ \), if and only if \( G' \) is even.
Outline

**Facility Location**
- Uncapcitated facility location (UFL)
- Prize-collecting facility location (PFL)

**Dual integrality**
- Characterization and algorithm for PFL
- Characterization and algorithm for UFL

**Proofs**
- Total dual integrality of PFL
- Total dual integrality of UFL
Given a $(0, \pm 1)$ matrix $A = (a_{ij})_{p \times q}$, it is associated with a bipartite digraph $D$ with vertex set $\{r_1, r_2, \ldots, r_p\} \cup \{c_1, c_2, \ldots, c_q\}$ s.t.

- $c_j r_i$ is an arc in $D$ if $a_{ij} = 1$; and
- $r_i c_j$ is an arc in $D$ if $a_{ij} = -1$,

For each cycle $C$ of $D$ (not necessarily necessarily directed), define

$$\rho(C) = (-1)^{|C|/2} \Pi \{a_{ij} | r_i c_j \in C\}$$

**Definition (Yannakakis, 1985)**

Matrix $A$ is *restricted totally unimodular* (RTU) if $\rho(C) = 1$ for each cycle $C$ of $D$
Restricted Totally Unimodular Matrix

Given a \((0, \pm 1)\) matrix \(A = (a_{ij})_{p \times q}\), it is associated with a bipartite digraph \(D\) with vertex set \(\{r_1, r_2, \ldots, r_p\} \cup \{c_1, c_2, \ldots, c_q\}\) s.t.
- \(c_jr_i\) is an arc in \(D\) if \(a_{ij} = 1\); and
- \(r_ic_j\) is an arc in \(D\) if \(a_{ij} = -1\),

For each cycle \(C\) of \(D\) (not necessarily directed), define

\[
\rho(C) = (-1)^{|C|/2} \prod\{a_{ij} | r_ic_j \in C\}
\]

**Definition (Yannakakis, 1985)**
Matrix \(A\) is **restricted totally unimodular (RTU)** if \(\rho(C) = 1\) for each cycle \(C\) of \(D\)

**Lemma (C, Chen, Zang, 2012)**
Let \(\omega(C) = \sum_{e \in C} a_e\). Then \(\rho(C) = 1 \iff \omega(C) \equiv 0 \pmod{4}\).
Even graph $\Rightarrow$ TDI-ness of PFL

- Even digraph $G$ $\rightarrow$ bipartite digraph $H$ such that $\omega(C) \equiv 0 \pmod{4}$ for every cycle $C$ in $H$
- $H$ is the digraph associated with the constraint matrix $M$ of $(PP')$
- Maximize $\sum_{uv \in A} w_{uv} x_{uv} + \sum_{v \in V} w_v y_v$
- subject to $\sum_{uv \in A} x_{uv} + y_v \leq 1 \forall v \in V$,
  $-x_{uv} + y_u \geq 0 \forall uv \in A$,
  $x_{uv} \geq 0 \forall uv \in A$,
  $y_v \geq 0 \forall v \in V$.
- $M$ is RTU $\Rightarrow$ Find an integral optimal solution $(x^*, y^*)$ of $(PP')$ combinatorially and efficiently [Yannakakis, 1985]
- $(x^*, y^*)$ is also an optimal solution to $(PP)$
- $M$ is totally unimodular $\Rightarrow$ The constraint matrix of $(PP)$ is totally unimodular $\Rightarrow$ TDI-ness

Xujin Chen, Zhibin Chen, Wenan Zang

Total Dual Integrality in Facility Location
Even graph $G \rightarrow$ bipartite digraph $H$ such that
- $\omega(C) \equiv 0 \pmod{4}$ for every cycle $C$ in $H$
- $H$ is the digraph associated with the constraint matrix $M$ of

(PP') Maximize $\sum_{uv \in A} w_{uv} x_{uv} + \sum_{v \in V} w_v y_v$
subject to $\sum_{uv \in A} x_{uv} + y_v \leq 1 \quad \forall v \in V,$
$-x_{uv} + y_u \geq 0 \quad \forall uv \in A,$
$x_{uv} \geq 0 \quad \forall uv \in A,$
$y_v \geq 0 \quad \forall v \in V.$
Even digraph $G \rightarrow$ bipartite digraph $H$ such that

- $\omega(C) \equiv 0 \pmod{4}$ for every cycle $C$ in $H$
- $H$ is the digraph associated with the constraint matrix $M$ of

$$(PP') \quad \begin{array}{c}
\text{Maximize} \\
\text{subject to}
\end{array} \quad \sum_{uv \in A} w_{uv}x_{uv} + \sum_{v \in V} w_vy_v$$

$$\sum_{uv \in A} x_{uv} + y_v \leq 1 \quad \forall v \in V,$$

$$-x_{uv} + y_u \geq 0 \quad \forall uv \in A,$$

$$x_{uv} \geq 0 \quad \forall uv \in A,$$

$$y_v \geq 0 \quad \forall v \in V.$$ 

- $M$ is RTU $\Rightarrow$ Find an integral optimal solution $(x^*, y^*)$ of $(PP')$ combinatorially and efficiently [Yannakakis, 1985]
Even graph $\Rightarrow$ TDI-ness of PFL

Even digraph $G \rightarrow$ bipartite digraph $H$ such that

- $\omega(C) \equiv 0 \pmod{4}$ for every cycle $C$ in $H$
- $H$ is the digraph associated with the constraint matrix $M$ of

\[
(PP') \quad \text{Maximize} \quad \sum_{uv \in A} w_{uv}x_{uv} + \sum_{v \in V} w_v y_v
\]
subject to
\[
\sum_{uv \in A} x_{uv} + y_v \leq 1 \quad \forall v \in V,
\]
\[
-x_{uv} + y_u \geq 0 \quad \forall uv \in A,
\]
\[
x_{uv} \geq 0 \quad \forall uv \in A,
\]
\[
y_v \geq 0 \quad \forall v \in V.
\]

- $M$ is RTU $\Rightarrow$ Find an integral optimal solution $(x^*, y^*)$ of $(PP')$ combinatorially and efficiently [Yannakakis, 1985]
- $(x^*, y^*)$ is also an optimal solution to $(PP)$
Even graph \( G \Rightarrow \) TDI-ness of PFL

Even digraph \( G \rightarrow \) bipartite digraph \( H \) such that

- \( \omega(C) \equiv 0 \pmod{4} \) for every cycle \( C \) in \( H \)
- \( H \) is the digraph associated with the constraint matrix \( M \) of

\[
\begin{align*}
(PP') & \quad \text{Maximize} & & \sum_{uv \in A} w_{uv} x_{uv} + \sum_{v \in V} w_v y_v \\
\text{subject to} & & \sum_{uv \in A} x_{uv} + y_v & \leq 1 \quad \forall v \in V, \\
& & -x_{uv} + y_u & \geq 0 \quad \forall uv \in A, \\
& & x_{uv} & \geq 0 \quad \forall uv \in A, \\
& & y_v & \geq 0 \quad \forall v \in V.
\end{align*}
\]

- \( M \) is RTU \( \Rightarrow \) Find an integral optimal solution \((x^*, y^*)\) of \((PP')\) combinatorially and efficiently [Yannakakis, 1985]
- \((x^*, y^*)\) is also an optimal solution to \((PP)\)
- \( M \) is totally unimodular \( \Rightarrow \) The constraint matrix of \((PP)\) is totally unimodular \( \Rightarrow \) TDI-ness
We are given $G = (F \cup C, A)$ and $c \in \mathbb{Z}_{F \cup A}^+$ such that $G'$ is even.

Case 1. $G = G'$: Similar
Case 2. $G \neq G'$:
We are given $G = (F \cup C, A)$ and $c \in \mathbb{Z}_{\geq 0}^{F \cup A}$ such that $G'$ is even

**Case 1.** $G = G'$: Similar

**Case 2.** $G \neq G'$:

- Add some new facilities and arcs to $G' \rightarrow$ bipartite digraph $G''$
  with cost $c''$
Even graph ⇒ TDI-ness of UFL

We are given $G = (F \cup C, A)$ and $c \in \mathbb{Z}_{+}^{F \cup A}$ such that $G'$ is even

Case 1. $G = G'$: Similar

Case 2. $G \neq G'$:

- Add some new facilities and arcs to $G' \rightarrow$ bipartite digraph $G''$ with cost $c''$

- The constraint matrix of UFL system on $G'$ is RTU ⇒ integral optimal solutions to the primal and dual on $(G'', c'')$
We are given $G = (F \cup C, A)$ and $c \in \mathbb{Z}_{+}^{F \cup A}$ such that $G'$ is even.

**Case 1.** $G = G'$: Similar

**Case 2.** $G \neq G'$:

- Add some new facilities and arcs to $G' \rightarrow$ bipartite digraph $G''$ with cost $c''$
- The constraint matrix of UFL system on $G'$ is RTU $\Rightarrow$ integral optimal solutions to the primal and dual on $(G'', c'')$
- Transform the integral optimal solutions to be the ones for $(G, c)$
Even graph \( \Rightarrow \) TDI-ness of UFL

We are given \( G = (F \cup C, A) \) and \( c \in \mathbb{Z}_{+}^{F \cup A} \) such that \( G' \) is even

**Case 1.** \( G = G' \): Similar

**Case 2.** \( G \neq G' \):

- Add some new facilities and arcs to \( G' \rightarrow \) bipartite digraph \( G'' \) with cost \( c'' \)
- The constraint matrix of UFL system on \( G' \) is RTU \( \Rightarrow \) integral optimal solutions to the primal and dual on \( (G'', c'') \)
- Transform the integral optimal solutions to be the ones for \( (G, c) \)
- The arbitrary choice of \( c \Rightarrow \) combinatorial algorithms & TDI-ness
Facility Location  Dual integrality  Proofs

Total dual integrality of PFL  Total dual integrality of UFL

Xujin Chen, Zhibin Chen, Wenan Zang

Total Dual Integrality in Facility Location